

GEF-Th-17/1993
hep-ph/9311219

FERMIONS, ANOMALY AND UNITARITY IN HIGH-ENERGY ELECTROWEAK INTERACTIONS

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ABSTRACT

We report the "state of the art" of the problem of $B + L$ violation in high-energy electroweak scatterings. Results of various analyses point toward (though do not prove rigorously yet) the "half-suppression", i.e., that the $B + L$ violating cross section remains suppressed at least by the negative exponent of the single instanton action, at all energies. Most interesting techniques developed in this field are reviewed. Particular attention is paid to unitarity constraints on the anomalous cross section, and to some conceptual problem involving the use of the optical theorem in the presence of instantons.

1. Introduction

We first present an overview of the progress made in the last few years in the problem of $B + L$ violation in high energy electroweak processes. The earlier part of the development will then be reviewed, which serves as a technical introduction to the subsequent sections.

1.1. Overview

In the standard electroweak theory, the baryon (and lepton-) number is not strictly conserved¹ as a consequence of the chiral anomaly²

$$\partial_\mu J^\mu = \frac{g^2}{16\pi^2} \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \quad (1)$$

where

$$J^\mu \equiv \bar{\psi}_L \gamma^\mu \psi_L \quad (2)$$

is the chiral current for a lefthanded doublet ψ_L in the theory. Due to dynamical, non perturbative effects (such as instantons or sphalerons) this leads to physical processes with the selection rule,

$$\Delta(B + L) = (3 + 3)\Delta\mathcal{N}_{CS}; \quad \Delta(B - L) = 0, \quad (3)$$

where \mathcal{N}_{CS} is the Chern Simons number (see Eq.(178)). For instance, an instanton leads to elementary processes such as

$$q_L + q_L \rightarrow 7\bar{q}_L + 3\bar{l}_L + n_w W + n_z Z + n_h H. \quad (4)$$

The instanton describes a transition between the neighboring classical vacua^{3,4}. It is well known that in the standard electroweak model the barrier height corresponds to the energy of the sphaleron configuration⁵,

$$E_{sp} \equiv \frac{\pi}{\alpha} M_W \sim 10 \text{ TeV}, \quad (5)$$

(where $\alpha \equiv \alpha_W \sim 1/32$.) Therefore one expects that at energies much lower than E_{sp} the cross sections for such $\Delta(B+L) \neq 0$ processes are typically suppressed by the t'Hooft's tunnelling factor,

$$e^{-\frac{4\pi}{\alpha}} \sim 10^{-170} \quad (6)$$

utterly too small for them to be observable.

Nevertheless, in at least two situations, at high temperatures¹² and at very high fermion densities¹³, baryon number violation is believed to proceed unsuppressed. The interest and importance for such a possibility is mainly related to the problem of cosmological baryon number generation in the standard $SU(3) \times SU(2) \times U(1)$ theory of fundamental interactions. For a recent review, see e.g. Shaposhnikov²⁹.

Whether or not the $\Delta(B+L) \neq 0$ events will become observable at high energy *scattering processes* (at the energy available at SSC or LHC), has been the issue of an active debate for a few years now. From experimental physics point of view, the prospect of observing roughly isotropic production of a large number $n = O(\frac{1}{\alpha})$ of W or Z bosons is quite an exciting one⁸. From the theoretical point of view, the problem involves some of the most subtle aspects of non-Abelian gauge theories, such as chiral anomaly, degenerate classical vacua and quantum tunnelling among them, large order breakdown of perturbation theory, compatibility of semiclassical expansions with unitarity, and so on.

It is the purpose of the present paper to review the latest developments in this field of research and assess our general understanding. Earlier works have been fully reviewed by Mattis¹⁰. See also the proceedings of the Santa Fe workshop⁶. For a more recent review, see Tinyakov¹¹.

Historically, after first suggestions^{18,19}, the semi-classical estimate of the total cross section with $\Delta(B+L) \neq 0$ was shown^{20,21} to grow exponentially with energy (at low energies), stimulating further works²³⁻³¹. It was noted immediately³² that the growth could not continue indefinitely as the computed cross section violated the unitarity limit above the sphaleron mass energy E_{sp} . Quantum corrections around the instanton were studied³²⁻³⁴ and it was shown that all tree type corrections involving the final states (so called soft-soft corrections) contributed to the nontrivial, exponential energy dependence. In particular, it was argued³³ that these

corrections could be expressed in the form,

$$\sigma_{\Delta(B+L) \neq 0} = e^{\frac{4\pi}{\alpha} F(x)} \quad (7)$$

where the function $F(x)$ (called sometimes "Holy Grail" function) has an expansion in $x \equiv E/E_{sp} \ll 1$.

Two distinct approaches, the R term method^{35,37} and the valley method³⁹⁻⁴⁵, were developed to compute the anomalous cross section, and used in recent perturbative calculations⁵⁵⁻⁶⁰. Subsequently the equivalence of the two methods as regards the final state corrections, has been argued to hold to all orders of perturbations around the instanton⁵². The developments which followed, to be discussed in Sections 2 and 4, make crucial use of these methods.

A non perturbative model, utilizing the valley field and the optical theorem, was presented by Khoze and Ringwald⁴⁹, with the aim of resumming all final tree corrections.

Other kinds of corrections involving the initial, hard particles (the so called hard-hard and hard-soft corrections) were then studied⁶³⁻⁶⁷, following the impetus provided by the works of Mueller. The results suggest the exponentiation of loop contributions to all orders, however, in the direction of suppressing the $\Delta(B+L)$ cross section.

Such an exponentiation supports strongly the idea that the quantum corrections involving the initial hard particles can also be included in a modified semi-classical approximation. Led by this thought several different approaches have been proposed. Works reviewed in Section 2 may indeed be called "Search for the new semi-classical field", or perhaps, more poetically, "Search for the Holy Grail."

Mueller⁶⁹ and independently Mc Lerran et al.⁶⁸, propose a classical equation of motion for the fields, with the source term to take account of the initial energetic particles. They show that to lowest orders their solution automatically reproduces the quantum loop corrections found earlier by direct calculations. However, as pointed out by these authors themselves, the necessity of using Minkowskian time or complex time (and in general complex fields) and the consequent complexity of the field equation involved, seem to make the task rather a formidable one.

An alternative approach was proposed by Rubakov and Tinyakov⁷³⁻⁸⁰. They propose to study, instead of the original $2 \rightarrow \text{all}$ type cross sections, the processes involving initial many-body states, $n \rightarrow \text{all}$, $n = \frac{\nu}{g^2}$, in the limit $g^2 \rightarrow 0$, with fixed ν and fixed total initial energy E . The idea is that once a semi-classical approximation for such many-body processes has been established, one can then take the limit $\nu \rightarrow 0$, hopefully recovering the answer for the $2 \rightarrow \text{all}$ process. The coherent state formalism first introduced in the $\Delta(B+L) \neq 0$ problem by Khlebnikov and others³⁵ plays a powerful role in this approach. Latest work⁸⁴ seems to show that the above limit is indeed a smooth one.

It is however not clear at the moment whether these approaches can lead to a new truly semi-classical approximation and what the eventual answer might be.

Perhaps the clearest physical picture of what happens in high-energy $\Delta(B+L)$

$L) \neq 0$ processes, has been given by the study of quantum-mechanical analogue problems^{85–96}. There are two competing factors which determine the energy dependence of the cross section²⁷. One, the semi-classical tunneling factor, rapidly grows with energy, and the suppression is altogether lifted when the sphaleron energy is reached. Another factor is the probability amplitude - mismatch - between the most favorable state (for the purpose of tunnelling) and the initial two-particle state. And this factor (an analogue of Landau's semiclassical matrix elements) gets strongly damped as the energy increases. As a consequence the full amplitude never gets large, leading to the "half-suppression" result (see below). Diakonov and Petrov⁹⁴ recently applied to the problem a generalization⁹³ of such a WKB approximation to field theory, finding some indication for the suppression of the cross section for the isotropic production of many W 's, even at very high energies: $E \gg E_{sp}$.

Quite parallel to the developments mentioned above, several arguments, essentially all taking unitarity constraints into account, were presented^{103–114}, which suggested that the $\Delta(B + L) \neq 0$ cross section remains suppressed *at least* by

$$e^{-\frac{2\pi}{\alpha}} \quad (8)$$

(so-called "half suppression"). In spite of their apparent differences, these arguments are actually closely interrelated, and depend only on the S-wave unitarity and the dominance of multiparticle production events.

The whole question was analysed from a somewhat different angle by the present authors^{115,125}. The approach using the optical theorem and the valley field to compute the total anomalous cross section⁴⁹, takes unitarity into account automatically, and furthermore seems to enable one to compute the Holy Grail function up to the sphaleron energy. However, a new kind of problem arises (which may be termed the "unitarity puzzle"). Namely, how can one extract the part of the imaginary part of the elastic amplitude, that corresponds to the $\Delta(B + L) \neq 0$ intermediate states? In other words, how is unitarity satisfied in the presence of topologically nontrivial effects such as instantons? A partial answer was given by the equivalence proof by Arnold and Mattis⁵²; however the crucial issue concerns the initial particles, see Section 5. Or, when does an "instanton-anti-instanton" type configuration cease to be topologically nontrivial? To clarify these issues requires a detailed study of the behavior of chiral fermions in a background of instanton anti-instanton type (hence the title of this review!). The result of this investigation leads us once more to the above mentioned "half-suppression" of the $\Delta(B + L) \neq 0$ cross section.

The rest of the paper is organized as follows. In subsections 1.2, 1.3 and 1.4 the earlier developments, including the original instanton calculation, the R -term method and the valley method, are reviewed, which also serve for fixing the convention and notation for later sections. The reader already involved in the research in this field may well skip this section; for others this section should provide an appropriate technical introduction.

The Section 2 contains a somewhat detailed review of more recent, new semi-classical approaches to the Holy Grail function. The discussion is divided into two parts, the first (subsection 2.1) dealing with attempts to take into account the effects of the initial high energy particles into a semi-classical equation, the second (subsection 2.2) being mainly concerned with the multi-particle approach of Rubakov and Tinyakov.

In Section 3 we review those studies based on quantum mechanical analogue problems, which appear to provide an intuitive understanding of the whole problem.

In Section 4 various analyses, leading to unitarity bounds on the baryon number violation in high energy electroweak scatterings, are reviewed. We first discuss a simple, multi-instanton unitarization picture and the general unitarity bound following from the S-wave dominance (subsection 4.1). The results are corroborated by an explicit resummation of multi-instanton contributions in subsection 4.2. The physical difference between the high *temperature* or high *density* transitions and the high *energy* $B + L$ violation is briefly mentioned in subsection 4.3.

The calculation of the anomalous cross section via optical theorem, done with the valley method, is critically analysed in Section 5. After the discussion of the unitarity puzzle (subsection 5.1), the fermion Green function in the background of the valley field is studied and the anomalous part of the forward elastic amplitude identified (subsection 5.2). The melting of the instanton anti-instanton pair, and the ensuing transition to purely perturbative amplitude, is discussed in subsections 5.3 (through the study of the Chern-Simons number) and in 5.4 (in which the fermion level crossing in the valley is analysed). The implication of these results to high energy electroweak processes is summarized in subsection 5.5.

We conclude (Section 6) by discussing a unifying and consistent picture which seems to emerge through different types of analyses reviewed here.

1.2. The original instanton calculation

In this section we recall briefly the calculation of Ringwald²⁰ and Espinosa²¹ for the cross section of the process (4). See Mattis¹⁰ for more details. In order to keep formulas as simple as possible we shall take only fermions and W bosons as external particles. Let us consider the Euclidean $n + N_F$ point Green function (N_F being the number of the lefthanded fermion doublets),

$$G(x_1, \dots, x_n, y_1, \dots, y_{N_F}) = \int dA d\psi d\bar{\psi} d\phi A_{\mu_1}(x_1) \dots A_{\mu_n}(x_n) \psi(y_1) \dots \psi(y_{N_F}) e^{-S(\phi, A_\mu, \psi, \bar{\psi})}. \quad (9)$$

This Green function gets contributions only from gauge fields with unit topological number $Q = 1$, where

$$Q \equiv \int d^4x \frac{g^2}{16\pi^2} \text{Tr} F_{\mu\nu} \tilde{F}_{\mu\nu} = \mathcal{N}_{CS}(-\infty) - \mathcal{N}_{CS}(\infty), \quad (10)$$

(\mathcal{N}_{CS} is the Chern Simons number, see Eq.(178)). In the semi-classical approximation one needs appropriate classical solutions with a finite action. In the electroweak theory one can show that no classical solutions exist in the presence of the Higgs vacuum expectation value. Indeed if one scales

$$\phi(x) \rightarrow \phi(ax); \quad A_\mu(x) \rightarrow aA_\mu(ax), \quad (11)$$

(which is a particular kind of variation), and considers the action as a function of a , one sees that no minimum exists for $a \neq 0$, for any functions ϕ and A . On the other hand the existence of a classical solution would require the action to be a minimum under *all* variations around it.

One is forced to look for a constrained instanton solution²². Although the solution is not known in a closed form, one knows asymptotic forms when the distance from the center of the instanton x_i is much less or greater than the size ρ of the instanton. Very near the instanton center it is given by

$$A_\mu^{(inst)}(x) \simeq -\frac{i}{g}U(\sigma_\mu\bar{\sigma}_\nu - \delta_{\mu\nu})U^\dagger \frac{(x-x_i)_\nu\rho^2}{(x-x_i)^2((x-x_i)^2 + \rho^2)},$$

$$\phi^{(inst)}(x) \simeq U \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \left(\frac{(x-x_i)^2}{((x-x_i)^2 + \rho^2)} \right)^{1/2}, \quad \frac{(x-x_i)^2}{\rho^2} \ll 1; \quad (12)$$

where $\sigma_\mu \equiv (i, \vec{\sigma})$, $\bar{\sigma}_\mu \equiv (\sigma_\mu)^\dagger$ and U is the global $SU(2)$ rotation. Far away from the center $A_\mu^{(inst)}(x)$ is proportional to a massive boson propagator; $\phi^{(inst)}(x)$ approaches a constant, $\phi^{(inst)}(x) \simeq U \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$.

The leading order approximation consists in keeping only up to the quadratic part of the fluctuations in the action and in substituting the fields by the classical ones in the pre-exponential factors. For fermions the "classical solutions" are the zero modes of the Dirac operator in the classical background, which behave as a free massless propagator asymptotically. The gauge field satisfies the massive free field equation far from the instanton center: its Fourier transform displays a pole at $k^2 = M_W^2$. Through the LSZ procedure, one gets the residue of this pole equal to

$$\mathcal{R}_\mu^a(\rho, \mathbf{k}) = \frac{4\pi^2 i}{g} \rho^2 \bar{\eta}_{\mu\nu}^a k_\nu, \quad (13)$$

where $\bar{\eta}$ is the usual t'Hooft symbol¹. Each fermion doublet contributes with a factor $\sim e^{ikx_i} \rho$, coming from the LSZ amputation of the zero mode,

$$\psi_0(x) \sim \frac{\rho}{(x-x_i)^3} \quad \text{for } x \rightarrow \infty. \quad (14)$$

The action evaluated at the classical field, is equal to²²

$$S_c = \frac{2\pi}{\alpha} + \pi^2 \rho^2 v^2 + O(\rho^4 v^2 M_H^2 \log(M_H \rho)). \quad (15)$$

Putting all these pieces together, and integrating over the collective coordinates (the instanton position, size and the isospin orientation U) one finds

$$A_{2 \rightarrow n} \sim n! \left(\frac{1}{v^2 g} \right)^n \exp(-S_{inst}) | \mathbf{k}_1 | \dots | \mathbf{k}_{n+2} | \quad (16)$$

where $S_{inst} \equiv 2\pi/\alpha$ and $n!$ comes from the integration over the size,

$$\int_0^\infty d\rho \rho^{2n} e^{-\pi^2 \rho^2 v^2}. \quad (17)$$

This gives the S matrix element at fixed n . The total cross section is found by squaring it and summing over the final states. To understand qualitatively its behavior¹⁰, recall that the relativistic phase space goes as $\bar{E}^{2n}/n!$; the squared amplitude as $(n!)^2 \bar{E}^{2n}$, where $\bar{E} \sim E/n$ is the average momentum. For large n these together give rise to an "exponential" series $\sum_n (E^{4/3})^{3n}/(3n)!$. A more careful estimation^{32,34,35} leads to

$$\begin{aligned} \sigma_{2 \rightarrow \text{all}} &\propto \frac{1}{s} \sum_n \frac{1}{(n!)^3} \left(\frac{3s^2}{8g^2\pi^2 v^4} \right)^n e^{-2S_{inst}} \\ &\sim \frac{1}{s} e^{\frac{4\pi}{\alpha}(-1 + \frac{9}{8}(\frac{E}{E_0})^{\frac{4}{3}})}, \end{aligned} \quad (18)$$

where $E_0 \equiv \sqrt{6}\pi M_W/\alpha$.

The cross-section grows exponentially with energy. If such a growth should continue up to and above the sphaleron energy,

$$E_{sp} \equiv \frac{\pi}{\alpha} M_W, \quad (19)$$

the exponential suppression factor would be compensated altogether. Actually, the approximations leading to Eq.(18) are valid for energies much less than the sphaleron energy as will be seen below.

A simple calculation shows that the cross section is dominated by the production of W bosons, with the average number

$$n \sim \frac{1}{\alpha} x^{4/3}, \quad (20)$$

where the definition

$$x \equiv \frac{E}{E_{sp}} \quad (21)$$

will be used throughout this review. x here must be small for these approximate estimates to be valid, but should not be too small either, so that the average multiplicity (20) is sufficiently larger than unity. Eq.(20) then implies that the mean energy of the final particles is $E/n \sim M_W x^{-1/3}$, showing that the final state particles carry soft momenta, while those in the initial ones hard (large) momenta.

The inclusion of the Higgs particles does not change these features in any essential manner.

Eq.(18), if extrapolated above the sphaleron energy, violates the unitarity limit. It must therefore be substantially corrected before the initial energy reaches E_{sp} . Does the $\Delta(B+L) \neq 0$ cross section nonetheless grow up to the point of saturating the unitarity limit, i.e., to a geometrical size, $\sim 1/s$? We are interested here in those corrections which exponentiate and modify the energy dependence in the exponent of the result, i.e., those which contribute to

$$F(x) = \lim_{g \rightarrow 0} \log \sigma_{2 \rightarrow \text{all}}. \quad (22)$$

F is called the Holy Grail function.

The corrections to the leading instanton result, are often classified into three groups (see Fig. 1): "soft-soft", "hard-hard" and "soft-hard" corrections. The first type involves only the final particles, the second one the initial particles and the third one both the initial and final particles. Also, contributions coming from other classical solutions with unit topological number such as multi-instanton configurations, could be important (see Section 4).

All the tree soft-soft corrections have been shown³³ to exponentiate and give rise to $1/\alpha$ terms in the exponent.^a In order to compute them, two distinct methods have been used: the valley method and the "R-term" method, which will be reviewed briefly in the next subsections.

1.3. The R-term method

In the R-term method, developed by Khlebnikov, Tinyakov and Rubakov³⁵, the summation over the final states is taken into account as a sort of correction to the action. This could be convenient for an application of the steepest descent method³⁸. We follow here an approach to the method developed by Mc Lerran¹⁰, which is simpler than the original one^{35,11}. The cross section is expressed as a double path integral, which within the perturbation theory is nothing but a compact way of implementing the standard Cutkovsky's rules.

Consider a theory with a scalar field ϕ with Minkowskian action

$$S = \int d^4x \left\{ \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi) \right\} \quad (23)$$

A Green's function (in momentum space) containing two initial particles and n final particles can be expressed as a path integral

$$G(p_1, p_2, k_1, \dots, k_n) = \int d\phi \phi(p_1)^* \phi(p_2)^* \phi(k_1) \dots \phi(k_n) e^{iS(\phi)}. \quad (24)$$

To get the total cross section one applies the LSZ procedure to the above, squares it (by doubling the fields) and sums over all possible final states. These operations

^aFor $x = O(1)$ the naive perturbation theory around the instanton would give an increasingly divergent series of type $\sim (\frac{1}{\alpha})^n$ which should necessarily be summed to all orders, i.e., nontrivial corrections to the Holy Grail function.

are compactly expressed¹⁰ by the introduction of

$$R(k) \equiv \lim_{k^2 \rightarrow m^2} (k^2 - m^2)^2 \phi(k) \phi^{*'}(k). \quad (25)$$

σ_{tot} can then be written as a double path integral:

$$\sigma_{tot} = \frac{1}{F} \int d\phi \int d\phi' R^*(p_1) R^*(p_2) e^{iS(\phi) - iS(\phi') + \int \frac{d^4 k}{(2\pi)^3} \delta(k^2 - m^2) \theta(k_0) R(k_0)}, \quad (26)$$

where F is the flux factor, $F \sim s$. Note that, thanks to the statistical factor $1/n!$ for identical particles, the factors associated with the final particles nicely exponentiate upon summation over n , and give rise to the third term in the exponent, the so-called R - term.

The R -term method is particularly powerful, when applied to the computation of the corrections to the leading semi-classical result, Eq.(18). We shall here content ourselves however to observe that to lowest order Eq.(26) reproduces Eq.(18).

In this approximation one splits the fields into the classical and quantum parts,

$$\phi(x) = \phi^{(inst)}(x; \alpha) + \phi^q(x), \quad (27)$$

where α indicates the ensemble of the collective coordinates, and substitutes the classical solution into the R - term, as well as in the ordinary actions and in the pre-exponents in (26). In the case of the standard model, the leading contribution to the exponent comes from W bosons and is represented in Fig. 2. Using Eq.(13) and by carefully continuing from Minkowskian to Euclidean space, one gets

$$\sigma_{tot} \propto \sum_{\mathbf{n}} | \langle \mathbf{n} | A | \mathbf{p}, -\mathbf{p} \rangle |^2 = e^{-16\pi^2/g^2} \int dt d\mathbf{x} d\rho d\rho' C e^W, \quad (28)$$

where (t, \mathbf{x}) is the Minkowski continued difference of the instanton positions, $x_i - x'_i$ and C contains the functional determinant over the nonzero modes, the Jacobian associated to the introduction of the collective coordinates as well as factors from fermion zero modes³⁵.

The crucial exponent W is given by

$$W = -iEt - \pi^2 v^2 (\rho^2 + \rho'^2) + \frac{\pi}{g^2} \rho^2 \rho'^2 \int \frac{d\mathbf{k}}{\omega_k} e^{ikx} (3\omega_k^2 + \mathbf{k}^2), \quad (29)$$

where $\omega_k \equiv \sqrt{\mathbf{k}^2}$.

Integration over \mathbf{x} by the saddle point method sets $\mathbf{x} = 0$, which leads to

$$W = -iEt - \pi^2 v^2 (\rho^2 + \rho'^2) + \frac{96\pi^2 \rho^2 \rho'^2}{g^2 t^4}. \quad (30)$$

The last integration over t and ρ, ρ' by the saddle point approximation yields the well known result, Eq.(18).

Higher-order corrections can be systematically taken into account by using the expansion Eq.(27) in Eq.(26). The resulting series in the Holy Grail function turns

out to be a power series in $x^{2/3}$ (see Eq.(39) below.) It means that in order to get a significant result for the energy of the order of E_{sp} ($x \sim 1$), one must be able to resum all the R -term contributions corresponding to soft tree graphs, or equivalently to solve the complete classical equations, including also the R -term.

It should be emphasized that the R -term method (and the R -term corrected classical equation) only makes sense in Minkowski space. For instance, by "instantons" one really means their analytic continuation to Minkowski space. Also, a new solution of R -term corrected classical equation with an $O(3)$ symmetry ("distorted instantons") can be found (though only in the linearized approximation)³⁸, but it turns out to be impossible to continue it to Euclidean space. As a result it is a nontrivial task to get a nonperturbative control of (even) the soft-soft part of the Holy Grail function (which we shall call $F^{ss}(x)$ to distinguish it from full $F(x)$).

The R -term method has been generalized by Espinosa³⁷, so as to take account of the effects of fermion pair production. His calculation shows that the inclusion of the fermionic R -term does not modify the result to exponential accuracy: it does not affect the Holy Grail function.

1.4. *The valley method*

An alternative approach to the calculation of the total cross section is based on the optical theorem^{32,34} which relates it to the imaginary part of the forward elastic amplitude. However since we are here interested only in the inclusive $\Delta(B+L) \neq 0$ cross section, not really the total cross section, a highly nontrivial problem arises in extracting the "anomalous piece" from the full imaginary part of the forward elastic amplitude. This issue will be discussed extensively in Section 5.

$\sigma_{\Delta(B+L) \neq 0}$ is given by

$$\sigma_{\Delta(B+L) \neq 0} = \frac{1}{s} Anom Im LSZ Wick \int dA d\psi d\bar{\psi} \psi \bar{\psi} \bar{\psi} \psi e^{-S(A) - \int \bar{\psi} \bar{D} \psi}, \quad (31)$$

where with *Anom*, *Im*, *LSZ*, *Wick* we indicated symbolically the operations of extracting the *anomalous* piece of the *imaginary part* of the on-shell amplitude (*LSZ procedure*), from the four point function *Wick continued* to Minkowskian space. One wishes to evaluate Eq.(31) by a semi-classical approximation. Since one deals here with an elastic amplitude, the relevant gauge background must belong to the trivial sector, with zero topological number. At the same time, however, it must describe nonperturbative effects of producing $\Delta(B+L) \neq 0$ intermediate states: it must be topologically nontrivial locally, as e.g., the instanton anti-instanton pair. As is well known from the example of a quantum mechanical double well, a simple sum of instanton and anti-instanton (at finite distances) is not an adequate background, since the expansion around it produces a large linear term in the "quantum" fluctuation, because is not a solution of field equations. The effort to minimize the latter term by a shift of the field introduces automatically an effective interaction between the instanton and anti-instanton.

The valley (or streamline) method provides a way to take such interactions into

account systematically. The valley method^{40–42} is a generalization of the standard Faddeev-Popov procedure of treating zero modes in quantum mechanics and in quantum field theory, to the case of quasi zero modes. If the action has a valley-like shape in the functional space, i.e. if its value slowly changes along the bottom of the valley (streamline), one must first perform Gaussian integrations in directions orthogonal to the streamline, then integrate the result along the streamline.

The valley trajectory (streamline) ϕ_α is a solution of the equation,^b

$$w(x, \alpha) \frac{\partial \phi_\alpha(x)}{\partial \alpha} = \frac{\delta S}{\delta \phi} |_{\phi=\phi_\alpha} \quad (32)$$

(w is an arbitrary weight function), α parametrizing the bottom of the valley. The generating functional Z in leading approximation is

$$Z \sim \int d\alpha \left\| \frac{\partial \phi_\alpha}{\partial \alpha} \right\|^2 e^{-S(\phi_\alpha)} \int d\phi \delta \left(\int d^4x (\phi - \phi_\alpha) \frac{\partial \phi}{\partial \alpha} w \right) e^{-\int (\phi - \phi_\alpha) \square_\alpha (\phi - \phi_\alpha)}. \quad (33)$$

This would be the same as the standard functional integration with a zero mode, were it not for the nontrivial integration over α .

In pure Yang-Mills theory in four dimensions, the valley equation was solved^{42–44} by using an Ansatz, $A_\mu = (2/g) \bar{\sigma}_{\mu\nu} (x_\nu/x^2) s(x^2)$. The problem is then reduced to that of a simple one-dimensional quantum mechanical double well, for which the solution to Eq.(32) is known. The valley trajectory $A_\mu^{(valley)}$ found this way (after a particular conformal transformation of the original ansatz) is

$$\begin{aligned} A_\mu^{(valley)} = & -\frac{i}{g} (\sigma_\mu \bar{\sigma}_\nu - \delta_{\mu\nu}) \left[\frac{(x - x_a)_\nu}{(x - x_a)^2 + \rho^2} \right. \\ & + \frac{(x - x_i)_\nu \rho^2}{(x - x_i)^2 ((x - x_i)^2 + \rho^2)} \\ & \left. + \frac{(x - x_i + y)_\nu}{(x - x_i + y)^2} - \frac{(x - x_i)_\nu}{(x - x_i)^2} \right] \end{aligned} \quad (34)$$

We defined

$$\begin{aligned} y^\mu & \equiv -R^\mu / (z - 1); \\ R^\mu & \equiv (x_i - x_a)^\mu; \\ z & \equiv (R^2 + 2\rho^2 + \sqrt{R^4 + 4\rho^2 R^2}) / 2\rho^2, \end{aligned} \quad (35)$$

where x_i^μ and x_a^μ are the centers of the instanton and anti-instanton, ρ is their (common) size. As is seen from Eq.(34) $A_\mu^{(valley)}$ interpolates between two solutions of the classical Yang-Mills equation: the simple sum of instanton and anti-instanton at infinite separation ($R = \infty$) and a gauge-equivalent of $A_\mu = 0$ (at $R = 0$).

For simplicity of writing, the size of the instanton and that of the anti-instanton will be taken to be equal here; no generality is however lost since the saddle point

^bThere is another possible way to define the valley trajectory⁴⁷ which is claimed to have some advantage over Eq.(32).

equations for ρ^a and ρ^i set them equal anyway, in the problem one is interested here.

The action of the valley is given^{43,44} by^c

$$S(A_\mu^{(valley)}) = \frac{48\pi^2}{g^2} \left[\frac{6z^2 - 14}{(z - 1/z)^2} - \frac{17}{3} - \log z \left(\frac{(z - 5/z)(z + 1/z)^2}{(z - 1/z)^3} - 1 \right) \right], \quad (36)$$

which asymptotically (as $R \rightarrow \infty$) behaves as

$$\frac{4\pi}{\alpha} - \frac{24\pi}{\alpha} \frac{\rho^4}{R^4} + O\left(\frac{1}{R^6}\right). \quad (37)$$

The first term is twice the instanton action, while the second one represents the attractive interaction between the instanton pair.

The behavior of the valley action as a function of R/ρ is plotted in Fig. 3.

In the case of the Weinberg-Salam theory, the solution of the valley equation is not known. At energies much lower than the sphaleron mass, however, one can justify^{32,34,49} the use of Eq.(34), Eq.(37) in Eq.(31). Furthermore the fermion fields ψ 's or $\bar{\psi}$'s in Eq.(31) may be replaced by the standard zero modes $\psi_0^{(a)}$'s (in the anti-instanton background) or $\bar{\psi}_0^{(i)}$'s (in the instanton background), respectively, see subsection 5.2. In this manner the result Eq.(18), is reproduced by the valley method, confirming once more the exponential growth of $\sigma_{\Delta(B+L) \neq 0}$ at low energies.

The equivalence between the R -term method and the valley approach has been derived *perturbatively* to all orders in x (or equivalently in $\frac{\rho}{R}$) in a rather formal way⁵²; their proof however neglects incoming particles (see Section 5) and also skips the problem of weight dependence of the result (see Section 2.1).

Without knowing the solution of the complete valley equation, the utility of the valley method in the electroweak theory is limited, unfortunately. Nonetheless, Khoze and Ringwald⁴⁹ attempted a nonperturbative calculation of the Holy Grail function by doing the following simplifications or assumptions (justified or not!). They *(i)* just add to the valley action Eq.(36) the Higgs contribution $-2\pi^2\rho^2v^2$; *(ii)* substitute the fermion fields by the standard fermion zero modes $\psi_0^{(a)}$'s and $\bar{\psi}_0^{(i)}$'s; and *(iii)* evaluate the resulting integrations

$$\sigma = Im \int dR d\rho \exp(ER - 2\pi^2\rho^2v^2 - S^{valley}(z)), \quad (38)$$

by the saddle point method (note that they commuted the Wick rotation with Im). The saddle point equation relates the relevant values of ρ and R to the initial energy E .

They find that the Holy Grail function increases monotonically and reaches precisely zero value (hence no exponential suppression of baryon number violation) at an energy of order of the sphaleron energy! ($x = x_{KR} \equiv 8\sqrt{3/5}$). See Fig. 4.

^cThe fact that the action depends on the valley parameters only through z is a reflection of the conformal invariance of the classical Yang-Mills equation.

This result is repeatedly referred to in the literature as a very encouraging sign that the $\Delta(B + L) \neq 0$ cross section might reach the geometrical size at high energies: after all, if a *reasonable* dynamical model gives an interesting result, why could it not be true also in the real world? Unfortunately, there are strong reasons to suspect that the assumption (ii) used by Khoze and Ringwald is too naive and is hardly justified, precisely for values of $R/\rho \leq 1$ where the valley action sharply drops to zero. We shall come back to the discussion of this model towards the end of Section 5.

Note that, by construction, the valley field depends on the choice of the weight function w : other choices might possibly introduce different kind of difficulties such as the "bifurcation" (i.e. the loss of the saddle point of Eq.(38))^{54,50,51}. A Minkowskian space formulation of the valley method was also given⁴⁵.

Finally, let us mention the recent perturbative calculations of the Holy Grail function.^{58–60} Balitsky and Schäfer⁵⁹ use the valley method (they check the result by using the effective Lagrangian approach) to compute the third term of the Holy Grail function, with the result:

$$F(x) = -1 + \frac{9}{8}\left(\frac{E}{E_0}\right)^{4/3} + \frac{3}{16}\left(\frac{E}{E_0}\right)^2 + \frac{3}{32}\left(4 - 3\frac{M_H^2}{M_W^2}\right)\left(\frac{E}{E_0}\right)^{8/3}\log\left(\frac{E_0}{E}\right). \quad (39)$$

where $E_0 \equiv \sqrt{6}E_{sp}$ so that $E/E_0 = x/\sqrt{6}$. Silvestrov instead uses the R - term method to recover the coefficient of $x^{8/3}\log(1/x)$, finding however only the piece (which agrees with Balitsky and Schäfer⁵⁸) depending on the Higgs and W boson masses. Diakonov and Polyakov⁵⁸ get the other piece, but with a factor 1/2 compared to Balitsky and Schäfer (these authors however work in the pure gauge sector).

2. In search of the Holy Grail.

In this section several recent attempts to compute nonperturbatively the full Holy Grail function will be reviewed in some details.

2.1. Initial corrections and attempts for a modified semi-classical approximation

Let us start with the discussion of corrections involving high-energy initial particles, i.e. , corrections to Eq.(18) due to interactions between hard particles or between hard and soft ones, (this definition can be applied indifferently both in the valley and R -term approaches). The lowest hard-hard quantum correction in the simplified case of bosonic particles is shown in Fig. 5, and hard-soft (hard-hard) corrections in the fermionic case in Fig. 1c (Fig. 1d).

First let us make some clarification on the terminology frequently used in the literature (hence adopted here too). The contribution to the amplitude of each type of correction ("hard-hard", "hard-soft" and "soft-soft") is not uniquely defined *by itself*; only the sum is well defined. The ambiguity essentially arises in both approaches from the arbitrariness of the choice of the scalar product (i.e. the

weight) with respect to which quantum fluctuations A^q are taken to be orthogonal to the streamline $\partial_\alpha A^{valley}$, see Eq.(32) (valley approach) or to the zero modes z_i of the second variation of the action around the instanton (in the R -term approach). This problem is often referred to simply as the "constraint dependence" of each type of contribution.

In fact, from the usual insertion of unity in the functional integral,

$$1 = J_w \int d\xi \delta\left(\int d^4x w(x) A^q(x) f(x)\right), \quad (40)$$

(where $f = \partial_\alpha A^{valley}$ or $f = z_i$ depending on the approach), a fictitious dependence on the weight is introduced. Such a dependence is expected to disappear order by order in perturbation theory only if all types of contributions are added together^{52,65}.

Single type of contributions thus in general depends on the choice of weight. Mueller⁶⁴ however argued that soft-soft terms are constraint-independent up to order x^2 . On the other hand, Khlebnikov and Tinyakov⁶⁵ proved (in the R -term approach) that, already at the $x^{8/3}$ order, the soft-soft contributions to the Holy Grail function, $F^{ss}(x)$, are ambiguous: they showed that an appropriate variation of the weight induces a variation of the corresponding Levine-Yaffe⁶² propagator of quantum fluctuations giving a modification of F^{ss} to this order. A similar conclusion were obtained in the valley approach by Arnold and Mattis⁵². Out of this observation comes also a suggestive idea⁶⁵ that if soft-soft corrections exponentiate but are ambiguous, perhaps the initial state corrections must also exponentiate to eliminate the ambiguity. As regards $O(x^{8/3})$ soft-soft term, the constraint dependence would be removed by hard-soft corrections only, since hard-hard ones are known to contribute starting from $O(x^{10/3})$ in the low energy expansion (see below).

Let us now concentrate on the study of initial-state corrections, considering the case of bosonic external particles in the one instanton sector, as often done in the literature. The leading order amplitude for a $2 \rightarrow n$ bosonic process can be written as

$$A_{2 \rightarrow n} = \int d\mu \mathcal{R}(p_1) \mathcal{R}(p_2) \prod_{j=1,n} \mathcal{R}(k_j) e^{-S_c} \quad (41)$$

where $d\mu$ stands for integrations over collective coordinates,

$$\mathcal{R}(p) \equiv \lim_{p^2 \rightarrow m^2} (p^2 - m^2) A^{inst}(p) \quad (42)$$

are the on-shell residues of the (Minkowski analytically continued) instanton field and p_1, p_2 are the hard momenta.

The first hard-hard correction comes from connecting together two hard particles with a propagator (while other soft lines are kept unchanged), see Fig. 5. It yields a correction

$$\delta A_{2 \rightarrow n} = \int d\mu D(p_1, p_2) \prod_{j=1,n} \mathcal{R}(k_j) e^{-S_c} \quad (43)$$

where

$$D(p, q) \equiv \lim_{p^2, q^2 \rightarrow m^2} (p^2 - m^2)(q^2 - m^2) G(p, q). \quad (44)$$

G is the Fourier transformed (Minkowski-continued) propagator of the boson fields in the instanton background (i.e. the constrained inverse of the second variation of the action evaluated at the instanton field, which we call \square_c). It satisfies the Levine-Yaffe equation⁶²:

$$\square_c G(x, y) = \delta(x - y) - \sum_{i,j} f_i (\Omega^{-1})_{ij} z_j \quad (45)$$

arising from constraining the quantum fluctuation to be orthogonal to general constraint f_i (having non singular overlap matrix $\Omega_{ij} = \int dx f_j(x) z_i(x)$ with the zero modes of \square_c , z_j). Clearly, for $f_i = z_i$ one recovers the usual BCCL propagator⁶¹.

We are thus interested in the behavior of the double residue of the propagator, $D(p, q)$, in the kinematical limit of interest $pq \rightarrow \infty$, and $p^2 = q^2 = m^2$. Explicit calculations in the $O(3)$ two dimensional sigma model⁷¹, and in pure $SU(2)$ gauge theory⁶³ yields

$$D(p, q) \rightarrow -cg^2 \rho^2(pq) \log(-pq) \mathcal{R}(p) \mathcal{R}(q) + O(g^2(pq)^0) \quad (46)$$

where R are the residues of the instanton fields, and c is a positive (model dependent) constant. Applying the Regge-pole technique to the operator \square_c , Voloshin⁶⁶ showed that the constant c is related to the translational zero modes of \square_c , $z_\mu \propto \partial_\mu A^{ist}$:

$$c = (z_\mu, |x|^2 z_\mu)^{-1}. \quad (47)$$

Due to the fast growth of $D(p, q)$ with the scalar product (pq) (Eq.(46)), this enhancement overcomes the suppression factor g^2 when $(pq) \sim (M_W/g^2)$. The correction Eq.(43) is no longer suppressed compared to the leading term (an estimate, $\rho \sim M_W$ is used).

In a subsequent work⁶⁴ Mueller furthermore showed by direct evaluation that, for a pure gauge theory, the quantum correction to the two (hard) point Green function up to the order α^N are of the form

$$\sum_{r=1}^N \frac{1}{r!} [(-cg^2 \rho^2(pq) \log(-pq))^r R(p) R(q) + O(g^{2r}(pq)^{r-1})] \quad (48)$$

where the leading terms of order $g^{2r}(pq)^r$ come from the "squared tree" diagrams of Figure 6. The dominance of the squared trees graph is justified by rapidity ordering arguments^{64,10}. For an alternative (simpler) derivation of this result see the work of Li et al.⁶⁷.

If one takes only the leading terms of Eq.(48) and sums the series up to $N = \infty$ the hard-hard terms exponentiate into a factor

$$e^{-cg^2 \rho^2(pq) \log(-pq)} \quad (49)$$

which gives a contribution of order $O(x^{10/3})$ to the Holy Grail function^d. This (naive) exponentiation could be taken as a first concrete hint suggesting the possibility of taking into account the initial corrections semi-classically.

The first attempt for a semi-classical treatment of initial corrections was done by Mattis, McLerran and Yaffe⁶⁸. Their idea consists of including the initial particles into an effective action S_{eff} by means of the identity:

$$\int dA A(p_1)A(p_2) \prod_{j=1,n} A(k_j) e^{iS(A)} = \int dA \prod_{j=1,n} A(k_j) e^{iS_{eff}(A)} \quad (50)$$

where

$$S_{eff}(A) \equiv S(A) - i \sum_{i=1}^2 \log A(p_i), \quad (51)$$

(the soft fields $A(k)$ can be treated with the usual R -term method). The solution of the (non covariant) field equation coming from the variation of S_{eff} ,

$$\frac{\delta S}{\delta A(x)} = i \sum_i e^{ip_i x} / A(p_i), \quad (52)$$

could in principle be used to give a semi-classical approximation that includes the leading initial effect for $x \sim 1$ ^e.

Another suggestion was made by Mueller⁶⁹. The main idea is to exploit the arbitrariness in the choice of the constraints for quantum fluctuations, (in the LY propagator) to simplify the form of initial state interactions as much as possible. He derived the behavior of the generic LY propagator in the kinematical (Minkowskian) region of interest (i.e. $(pq) \rightarrow \infty$ and on shell) and was able to find an appropriate (noncovariant) choice of the constraints that eliminates the leading term of $O(pq \log(-pq))$ of the LY propagator. This choice consequently eliminates all the leading multi-loop hard-hard corrections. As for the hard-soft interactions, Mueller demonstrated, by using a diagrammatic analysis that, for a single soft particle the interactions with hard particles can be included just by substituting the instantonic background by the solution W_ν^a of the (Minkowskian) Y.M. equation with a source term:

$$D_\mu^{ab}(W) G_{\mu\nu}^b(W) = J_\nu^a(W) - \frac{1}{\xi} D_\nu^{ac}(A) D_\mu^{cb}(A) W_\mu^b \quad (53)$$

where $D(W), D(A)$ are covariant derivatives respectively in W or in the instanton background, $G_{\mu\nu}^b(W)$ is the field strenght in terms of W and ξ is the usual gauge parameter. The source J contains in a complicated (nonlocal) way the space-time

^dNote that the condition of factorization $E/N \gg 1/\rho$ (high energies in each final branch of the tree graphs) and of exponentiation $N \gg g^2 \rho^2(pq)$ seems to force $x \ll 1$

^eAlso in this case, if the Minkowski propagator had a singular behavior, the loop contributions could be important. Approximating the solution by the usual instanton and doing perturbations around this (wrong) background, the authors found that a cancellation arises between potentially large one-loop terms and that tree contributions reproduce the Mueller's corrections: this suggests that in the correct background loops are negligible.

integral of the solution W , and has a singularity structure that makes the continuation to Euclidean time impossible. For more than one soft particles, hard effects are to be taken into account with the usual R -term scheme (all tree graphs should be summed) in the new background W . A Euclidean valley resummation does not seem to be applicable due to the intrinsically Minkowski singularities.

It is not clear at present how useful the two approaches mentioned here might be in practice, as they (both) involve the solution of an extremely difficult (non-covariant) Yang-Mills equation with a source depending in a complicated manner on the solution itself^f; they are furthermore incomplete in the sense that they must be complemented with some method for resumming soft trees. On the other hand, these innovating works perhaps tell us that a semi-classical approach including initial corrections is in principle possible.

2.2. Multiparticle approach

A more promising approach for estimating the complete Holy Grail function (i.e. including initial state corrections) is that of Rubakov and Tinyakov^{73,74}. The idea is to compute first the inclusive cross section $n_i \rightarrow \text{all}$ for large n_i ($n_i = \frac{\nu_i}{g^2}$), for which a semi-classical method is likely to be applicable straightforwardly (see below). Then ν_i can be sent to zero *afterward* in $F(x, \nu_i) = \frac{g^2}{16\pi^2} \log \sigma_{n_i \rightarrow \text{all}}$, assuming that the limit is smooth, $g^2 \log \sigma_2 \sim \lim_{\nu_i \rightarrow 0} F(x, \nu_i)$.

Note that while for σ_2 the exponentiation of quantum (loop) corrections would appear rather miraculous, it is quite reasonable that multiparticle inclusive cross section can be evaluated semi-classically (this will be proved below).

There are two ingredients for such semi-classical calculations. First, an appropriate Minkowski boundary conditions (taking account of the quantum number, energy, etc. of the initial state) must be used. This is to be compared to the standard calculation of perturbative S-matrix elements where the vacuum (Feynman) boundary condition is used. For this purpose the coherent state representation of the S-matrix turns out to be particularly suited.

Secondly, the fact that one is interested in nonperturbative, classically forbidden processes, forces one to consider the time evolution partially in Euclidean direction (see below). As a result the classical equation (and its solution) will be defined along a complex time contour.

For completeness, let us first take a few steps back and review briefly the coherent state representation for the S matrix (Khlebnikov et al.³⁵ have used this method extensively in formulating the R -term method.)

First consider the one-dimensional harmonic oscillator, (generalisation to many-degrees of freedom is straightforward). More details can be found in textbooks⁷⁰. In terms of the vacuum state $|0\rangle$ and creation operator A^\dagger , a (non-normalized)

^fOne must also check, in the context of baryon number violation, that the solution has nontrivial topological number.

coherent state can be defined, for every complex number a as

$$|a\rangle = e^{A^\dagger a} |0\rangle. \quad (54)$$

It is easy to derive from commutation relation $[A, A^\dagger] = 1$ the following properties:

$$A |a\rangle = a |a\rangle \quad (55)$$

$$\langle b|a\rangle = e^{b^* a} \quad (56)$$

$$e^{f A^\dagger A} |a\rangle = |a e^f\rangle. \quad (57)$$

The wave function of a coherent state in position representation is also useful:

$$\langle q|a\rangle = e^{-\frac{1}{2}a^2 - \frac{1}{2}\omega q^2 + \sqrt{2\omega} a q}, \quad (58)$$

(it can be checked that this satisfies Eq.(55) when A is expressed in the coordinate representation: $A = \frac{1}{\sqrt{2}}(\sqrt{\omega}q + \frac{1}{\sqrt{\omega}}\frac{\partial}{\partial q})$). The action of an operator S in the coherent state representation is obviously given by its matrix elements: defining the kernel $S(\bar{b}, a) \equiv \langle b|S|a\rangle$,

$$\langle b|S|\psi\rangle = \int \frac{da d\bar{a}}{2\pi i} e^{-\bar{a}a} S(\bar{b}, a) \psi(\bar{a}), \quad (59)$$

where the exponential takes into account the non-normalization of our coherent states. Note also that in the integral we must fix $\bar{a} = a^*$.

Finally note the following relation involving the matrix element of an operator S between fixed number of quanta and its kernel:

$$\langle n|S|m\rangle = \partial_a^m \partial_{\bar{b}}^n S(\bar{b}, a)|_{a=\bar{b}=0}. \quad (60)$$

That is all one needs to construct the coherent state representation of the scattering matrix (in the Fock's space of asymptotic states) ⁷⁰. Starting from the usual interaction representation of S

$$S = \lim_{T_i \rightarrow -\infty} \lim_{T_f \rightarrow \infty} e^{iH_0 T_f} U(T_f, T_i) e^{-iH_0 T_i}, \quad (H_0 = \int d\mathbf{k} \omega_{\mathbf{k}} A_{\mathbf{k}}^\dagger A_{\mathbf{k}}) \quad (61)$$

and introducing the resolution of unity in terms of fields at T_i, T_f (which allows to use the path integral representation for $\langle \phi_f | U(T_f, T_i) | \phi_i \rangle$), we obtain the following expression for the kernel of the S matrix:

$$S(\bar{b}, a) = \int d\phi_i d\phi_f d\phi e^{B_i(a_{\mathbf{k}}, \phi_i) + B_f(\bar{b}_{\mathbf{k}}, \phi_f)} e^{iS(\phi)} \quad (62)$$

where the boundary terms

$$\begin{aligned} B_i(a_{\mathbf{k}}, \phi_i) &\equiv -\frac{1}{2} \int d\mathbf{k} a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2i\omega_{\mathbf{k}} T_i} - \frac{1}{2} \int d\mathbf{k} \omega_{\mathbf{k}} \phi_i(\mathbf{k}) \phi_i(-\mathbf{k}) \\ &+ \int d\mathbf{k} \sqrt{2\omega_{\mathbf{k}}} \phi_i(-\mathbf{k}) a_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} T_i} \end{aligned} \quad (63)$$

$$\begin{aligned}
 B_f(\bar{b}_{\mathbf{k}}, \phi_f) &\equiv -\frac{1}{2} \int d\mathbf{k} \bar{b}_{\mathbf{k}} \bar{b}_{-\mathbf{k}} e^{2i\omega_{\mathbf{k}} T_f} - \frac{1}{2} \int d\mathbf{k} \omega_{\mathbf{k}} \phi_f(\mathbf{k}) \phi_f(-\mathbf{k}) \\
 &+ \int d\mathbf{k} \sqrt{2\omega_{\mathbf{k}}} \bar{b}_{\mathbf{k}} \phi_f(\mathbf{k}) e^{i\omega_{\mathbf{k}} T_f}
 \end{aligned} \tag{64}$$

come from wave functions of coherent states (see Eq.(58), and also Eq.(57)).

In the multiparticle approach of Rubakov and Tinyakov⁷³, one actually considers another, slightly different kernel: the idea is to project the initial coherent state onto the eigenstate of an operator

$$O = \int d\mathbf{k} f_{\mathbf{k}} A_{\mathbf{k}}^{\dagger} A_{\mathbf{k}} \tag{65}$$

(typically the energy) with a fixed eigenvalue Ω (generalization to cases with more than one operator is straightforward). One is then interested in:

$$S_{\Omega}(\bar{b}, a) \equiv \langle b | S P_{\Omega} | a \rangle. \tag{66}$$

Substitution in Eq.(66) of the integral representation of the projector

$$P_{\Omega} = \int d\xi e^{i(O-\Omega)\xi} \tag{67}$$

and use of property Eq.(57), yield

$$S_{\Omega}(\bar{b}, a) = \int d\xi \int d\phi_i d\phi_f d\phi e^{-i\Omega\xi} e^{B_i(a_{\mathbf{k}} e^{if_{\mathbf{k}}\xi}, \phi_i) + B_f(\bar{b}_{\mathbf{k}}, \phi_f)} e^{iS(\phi)} \tag{68}$$

where the function $f_{\mathbf{k}}$ refers to the operator O , Eq.(65).

A quantity of interest, then, is the transition rate for the microcanonical ensemble (respect to O). It is obtained by summing the inclusive cross sections over the initial states with a definite value of O ,

$$\sigma(\Omega) = \sum_{i,f} |\langle f | S P_{\Omega} | i \rangle|^2. \tag{69}$$

(Actually, there is a slight lack of precision in this expression. One is really interested here in a partial sum over B- (and L-) violating final states f , so that the unit operator part of the S-matrix actually drops out. Otherwise, Eq.(69) would have no physics contents! Formally, however, the sum over f is done without any restriction (see the next equation). The restriction over selected final states is to be taken into account by a judicious choice of the classical backgrounds.)

By substituting Eq.(68) in Eq.(69), one obtains:

$$\sigma(\Omega) = \int d\phi d\phi' da d\bar{a} db d\bar{b} d\xi e^W \tag{70}$$

where

$$\begin{aligned}
 W &= -i\Omega\xi - \int d\mathbf{k} \bar{b}_{\mathbf{k}} b_{\mathbf{k}} - \int d\mathbf{k} e^{-if_{\mathbf{k}}\xi} \bar{a}_{\mathbf{k}} a_{\mathbf{k}} + B_i(a_{\mathbf{k}}, \phi_i) + B_f(\bar{b}_{\mathbf{k}}, \phi_f) + iS(\phi) \\
 &+ B_i(a_{\mathbf{k}}, \phi'_i)^* + B_f(\bar{b}_{\mathbf{k}}, \phi'_f)^* - iS(\phi')^*.
 \end{aligned} \tag{71}$$

Note that a rescaling $a_{\mathbf{k}} \rightarrow a_{\mathbf{k}} e^{-if_{\mathbf{k}}\xi}$, $\bar{a}_{\mathbf{k}} \rightarrow \bar{a}_{\mathbf{k}} e^{if_{\mathbf{k}}\xi'}$, the integration over $d(\xi + \xi')$ as well as the substitution $\xi - \xi' \rightarrow \xi$ have been made.

The nice feature of $\sigma(\Omega)$ is that the functional integral appearing in Eq.(70), although very complicated, can be evaluated semi-classically in the limit of small g , provided that the external parameters Ω are of the order of $\frac{1}{g^2}$. Indeed, rescaling as

$$(\phi, \phi', \bar{a}, a, \bar{b}, b) \rightarrow (\phi, \phi', \bar{a}, a, \bar{b}, b)/g \quad (72)$$

one finds $W = \frac{\tilde{W}}{g^2}$, where \tilde{W} depends on the rescaled fields, on $g^2\Omega$ and on M_W , but not explicitly on g . Thus the usual saddle point technique can be applied, yielding a result of the form

$$\sigma(\Omega) = \exp\left\{\frac{1}{g^2}F(g^2\Omega, M_W)\right\}. \quad (73)$$

The saddle point field will be a solution of a classical equation of motion, with some new type of boundary conditions (different from those in the usual vacuum to vacuum transitions) involving the saddle point values of the complex fields a, b ; more on this later.

(The present formalism was actually introduced earlier by Khlebnikov, Tinyakov, Rubakov⁷², in which the operators O were chosen to be the four momentum. They considered the microcanonical ensemble of initial states with a fixed momentum $(E, \vec{0})$, with the idea that the new boundary conditions will be more physically relevant in processes of production of many particles, as they keep memory of the incoming and outgoing fields. The relevant saddle point configuration is an analytic continuation to an adequate complex time path of a real Euclidean time configuration, the so-called "periodic instanton". $\sigma(E)$ however turns out not to be directly related to the cross section σ_2 one is interested in⁷².)

The main new idea of Rubakov and Tinyakov⁷³ is to consider the cross section for the initial microcanonical ensemble with fixed energy E and fixed number of incoming particles $n_i = \nu_i/g^2$, $\sigma(E, n_i)$.

The advantage of this approach is that for $\nu_i \neq 0$ all the initial states corrections are automatically taken into account in the semi-classical approximation, since the initial particles are represented as fields appearing in the boundary terms. Thus, if σ_2 is recovered in the limit $\nu_i \rightarrow 0$, it is likely that the initial corrections are accounted for: this would reduce the problem of evaluating the Holy Grail function effectively to the search of an adequate classical configuration.

It was also shown⁷⁴ that, perturbing the functional integral for $\sigma(E, n_i)$ around the instanton (expansion valid for $E \ll E_{sp}$), that the leading hard-hard corrections thus inferred do coincide with the naively exponentiated Mueller's series, Eq.(49).

Let's us consider thus the expression Eq.(70) for $\sigma(E, n_i)$ (with the obvious generalizations: $d\xi \rightarrow d\xi d\eta$, $\Omega \rightarrow E, n_i$, $f_{\mathbf{k}} \rightarrow \omega_{\mathbf{k}}, 1$), and discuss the boundary conditions⁷⁵ that characterize the saddle point field.

The integration of the final states \bar{b}, b gives exactly a term $\delta(\phi_f - \phi'_f)$ (by completeness of coherent states); then the subsequent integration over ϕ_f, ϕ'_f (keeping

into account of all boundary terms[§]), gives a $\delta(\dot{\phi}_f - \dot{\phi}'_f)$. Thus, being $\phi = \phi'$, $\dot{\phi} = \dot{\phi}'$ at $T_f = +\infty$ the two classical solutions coincide over all Minkowskian space.

The integration over \bar{a}, a is Gaussian and can be done exactly; the integration over ϕ_i, ϕ'_i leads to the following boundary conditions at $T_i \rightarrow -\infty$ (the saddle point values for \bar{a}, a have been substituted already):

$$i\dot{\phi}_i(\mathbf{k}) + \omega_{\mathbf{k}}\phi_i(\mathbf{k}) = e^{i\Delta_{\mathbf{k}}}[i\dot{\phi}'_i(\mathbf{k}) + \omega_{\mathbf{k}}\phi'_i(\mathbf{k})] \quad (74)$$

$$i\dot{\phi}_i(\mathbf{k}) - \omega_{\mathbf{k}}\phi_i(\mathbf{k}) = e^{-i\Delta_{\mathbf{k}}}[i\dot{\phi}'_i(\mathbf{k}) - \omega_{\mathbf{k}}\phi'_i(\mathbf{k})] \quad (75)$$

where $\Delta_{\mathbf{k}} \equiv \omega_{\mathbf{k}}\xi + \eta$. Subsequent saddle point evaluation of the integral over ξ and η , will then fix the parameters of the solution in terms of E, n_i .

By time traslation invariance, the variable ξ associated with energy can always be taken as purely imaginary: $\xi = i\xi_0$; perturbative calculations^{74,75} suggest that also the saddle point value of η is purely imaginary; so we can choose $\eta = i\eta_0$. It was furthermore observed^{72,75} that the parameter ξ_0 can be removed from boundary conditions Eqs.(74-75) by choosing suitable contours in complex time plane: one choice could be the one shown in Fig. 7 (with $\xi_0 = BB'$).

If we assume the solution to be free at $\tau = \text{Re } t \rightarrow -\infty$, and write in this limit

$$\phi(\mathbf{k}) = f_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} + \bar{f}_{-\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \quad t \in \text{AB} \quad (76)$$

$$\phi'(\mathbf{k}) = g_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} + \bar{g}_{-\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \quad t \in \text{A'B'}, \quad (77)$$

substituting in Eqs.(74-75) we obtain simplified boundary conditions:

$$f_{\mathbf{k}} = e^{-\eta_0} g_{\mathbf{k}}; \quad \bar{f}_{\mathbf{k}} = e^{\eta_0} \bar{g}_{\mathbf{k}} \quad (78)$$

which relates asymptotic positive and negative frequency components on the two branches of the path, AB and A'B'.

As anticipated before, the necessity of using the complex-time deformed path comes out from the double requirement of having free particle incoming (outcoming) at Minkowskian time $\text{Re } t \rightarrow -(+)\infty$ and of describing a semi-classical, classically forbidden transition, that requires a part of the evolution to proceed in the Euclidean time.

The idea of the complex-time deformed paths is not really new: it is known (and was invented first) in quantum mechanics, in the W.K.B. approximation for the fixed energy Green's functions in the case of barrier penetration⁸³. In that case, the relevant classical solution is composed of Minkowski solutions for the initial and final parts of evolution and of an intermediate, Euclidean solution describing the tunnelling.

[§]Note that also a term

$$\frac{i}{2} \int d\mathbf{k} (\phi_{\mathbf{k}} \dot{\phi}_{-\mathbf{k}} - \phi'_{\mathbf{k}} \dot{\phi}'_{-\mathbf{k}})|_{T_i}^{T_f}$$

in the exponential coming from integration by parts of the kinetic term of the fields' action, are essential to the integration over initial and final fields.

Let us now have a closer look at the required properties of the classical field we are looking for. If the solution to field equations satisfying boundary conditions Eq.(78) is unique, then it must obey the relation $\phi(t^*) = \phi^*(t)$ (coming from reality of field equations); that implies the reality of the field on the positive Minkowski time. It follows from Eqs.(76-77) that

$$f_{\mathbf{k}} = \bar{g}_{\mathbf{k}}^*; \quad g_{\mathbf{k}} = \bar{f}_{\mathbf{k}}^*. \quad (79)$$

Also, it can be seen that the saddle point values of a, \bar{a} are complex conjugate of each other, which means that the sum over the initial states is dominated by a single coherent state.

Perturbative calculations^{72,75} support the conjecture that on the Euclidean time piece the solution might be real: this (together with the reality on the positive real time axis) would imply the presence of a turning point $\dot{\phi} = 0$ for $t = 0$ ⁷⁵.

All these hypotheses, put together, imply that the required saddle point configuration, determining the microcanonical cross section $\sigma(E, n_i)$ would be a solution of classical equations on the complex time path of Fig. 7 with boundary conditions:

$$\begin{aligned} \dot{\phi}|_{t=0} &= 0; \\ \phi(\mathbf{k}) &= f_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} + e^{\eta_0} f_{-\mathbf{k}}^* e^{i\omega_{\mathbf{k}}\tau}, \end{aligned} \quad (80)$$

where $\tau = \text{Re } t \rightarrow -\infty$.

Unfortunately, the "microcanonical" saddle point is not known at present even numerically at energies of order E_{sp} . A perturbative expression for this solution (valid at low energies) is known⁷⁵ in the context of two dimensional Abelian Higgs model: it looks like a chain of alternated instantons and anti-instantons. This study put into evidence the problem of the presence of singularities of analytically continued classical solutions in the complex time, that the chosen path must not touch. See the discussion below, Eq.(84).

A slight variation of the approach of Rubakov and Tinyakov, is to compute the cross section for a *given* coherent state of fixed energy

$$\sigma(E, \{a_{\mathbf{k}}\}) = \int d\phi d\phi' db d\bar{b} d\xi d\xi' e^W \quad (81)$$

with

$$\begin{aligned} W &= -iE(\xi - \xi') - \int d\mathbf{k} \bar{b}_{\mathbf{k}} b_{\mathbf{k}} + B_i(a_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\xi}, \phi_i) + B_f(\bar{b}_{\mathbf{k}}, \phi_f) + iS(\phi) \\ &+ B_i(a_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\xi'}, \phi_i')^* + B_f(\bar{b}_{\mathbf{k}}, \phi_f')^* - iS(\phi')^*, \end{aligned} \quad (82)$$

where now a, \bar{a} are the arbitrary but fixed complex numbers. Integration over final fields goes as before. What changes is the integration over ϕ_i, ϕ_i' that by saddle point evaluation give the initial boundary conditions:

$$\begin{aligned} i\dot{\phi}_i(\mathbf{k}) + \omega_{\mathbf{k}}\phi_i(\mathbf{k}) &= \sqrt{2\omega_{\mathbf{k}}} a_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}(T_i - \xi)} \\ i\dot{\phi}_i'(\mathbf{k}) - \omega_{\mathbf{k}}\phi_i'(\mathbf{k}) &= -\sqrt{2\omega_{\mathbf{k}}} \bar{a}_{-\mathbf{k}} e^{i\omega_{\mathbf{k}}(T_i - \xi')}. \end{aligned} \quad (83)$$

Clearly, if one would have known the solution of the boundary condition problem for *generic* values of a, \bar{a} one could estimate semi-classically Eq.(81) and recover directly the two particle cross section by differentiating twice $\sigma(E, \{a_{\mathbf{k}}\})$ (see Eq.(60)).

Unfortunately, the resolution of the general boundary condition problem is a hard task. Some recent works^{76,78,79} pursue a more modest goal of calculating semi-classically $\sigma(E, \{a_{\mathbf{k}}\})$ with a, \bar{a} fixed *a posteriori* to satisfy the boundary conditions Eq.(83), starting from some given classical solution. These works have many common features. They study exact solutions of field equations in conformally invariant models^h, obtained by some symmetry requirement that simplifies the problem to an equivalent one dimensional one. In particular, the first⁷⁶ studies the massless two dimensional $O(3)$ sigma model, searching for $O(2)$ symmetric solutions; the second⁷⁸ considers the $SO(4)$ conformally invariant solutions of Minkowskian $SU(2)$ pure Yang Mills theory, the so-called Lüscher-Schechter solutions^{81,82}; the third⁷⁹ studies an $O(4)$ invariant solution of a massless, four dimensional ϕ^4 theory with the positive ("right") sign of the coupling constant.

All these solutions are continued to a complex path as in Fig. 7, and possess a turning point at $t = 0$. It turns out that the position of the path relative to the singularities of the solution determines its very nature. As a significant example note that Yang Mills complex time solution is such that the Minkowskian action S and the topological charge Q evaluated along the path, obey the relation

$$\frac{g^2}{8\pi^2} \text{Im} S = Q = N, \quad (84)$$

where N is the number of singularities between the path and Minkowski time axis (relation that is identical to the usual multi-instanton). The path considered in his particular case corresponds to the one instanton sector.

Evaluating the average number of initial (from the derived a, \bar{a}) and final particles (from the saddle value of \bar{b}, b), it turns out that these solutions allow to compute the probability of a process with parametrically less particles in the initial state than the final one (for example⁷⁸ $\nu_f = \nu_i^{7/8}$ for small ν_i). But in the limit $\nu_i \rightarrow 0$ the cross section is exponentially damped by the full t'Hooft suppression factor $\exp\{-16\pi^2/g^2\}$. These solutions however do not maximize the transition probability in the one instanton sector at a given energy, (because the resulting a, \bar{a} have not opposite phase, see above). Thus these solution neither are directly related to the $2 \rightarrow$ all processes, nor can be used to establish a rigorous upper bound to that cross section. Nevertheless they are a useful benchmark to understand the property of more physically relevant solutions (such as the "microcanonical" one considered before).

An important, nontrivial question, is if general complex gauge fields configurations defined on a complex contour, satisfy a generalized Atiyah-Singer index

^hThey are all massless models that are supposed to describe the high-energy dynamics of the massive ones. Clearly with this approach it is impossible to recover the low-energy behavior of massive models.

theorem^{120,119} of the form:

$$n_l - n_r = \int_{\text{path}} \text{Tr} F \tilde{F} = Q \quad (85)$$

where n_l (n_r) are the left (right) handed zero modes of the Dirac operator. Such a theorem would imply that if we had a classical solution with complex topological charge $Q = \pm 1$, we would be assured the presence of the fermion zero mode which is essential for the nonvanishing of the anomalous cross section.

Also, one may ask if there is a level of the Dirac Hamiltonian crossing zero in time evolution (physically interpretable as creation or annihilation of a hole or a particle depending on the direction, see Section 5), in one-to-one correspondence with the zero modes. A step in the direction of answering these questions, was taken in a work by Rubakov and Semikoz⁷⁷, where the authors verified these conjectures in the case of the two dimensional Abelian Higgs model.

Let us end this section by a brief mention to a recent work of Mueller⁸⁴. Instead of the "microcanonical" cross section $\sigma(E, \nu_i)$ of Rubakov and Tinyakov, the author considers a slight variation, with the initial state consisting of the (equally weighted) sum of all pure coherent states with energy E and with $n_1 = \nu_1/g^2$ right moving and $n_2 = \nu_2/g^2$ left moving W's (he considers only the pure gauge sector). The processes with such modified initial states seem to be considerably closer kinematically to those with the two particle initial states, than the simple microcanonical ones. Subsequently the functional integral is expanded perturbatively around the single instanton. An accurate study of the hard-hard corrections, (up to $x^{10/3}$ in F), and of hard-soft (up to $x^{16/3}$) shows that in the limit $\nu_1 = \nu_2 = \nu_i \rightarrow 0$ these contributions to the exponent $F(x, \nu_1, \nu_2)$ join smoothly with the result for two particle initial state. This is quite nontrivial, and holds only after compensations of divergent $1/\nu$ terms.

It would thus appear that a semi-classical (perhaps numerical) study of processes of the form $n_1 + n_2 \rightarrow \text{all}$ provides a good way to estimate nonperturbatively the Holy Grail function for the desired $2 \rightarrow \text{all}$ transition. How realistic this possibility is in practice, is however unknown.

3. Quantum mechanical analogue problems

In order to get an intuitive, physical understanding of our four dimensional problem, several authors studied toy models in the context of one-dimensional quantum mechanics⁸⁵⁻⁹⁴. The results of these studies will be reviewed briefly in this section, following mainly the treatment of Diakonov and Petrov⁹⁴.

Consider a quantum mechanical double well coupled to a weak, rapidly oscillating field. The Lagrangian is given by

$$L = \frac{1}{g^2} \left(\frac{1}{2} \left(\frac{dq}{dt} \right)^2 - \frac{1}{8} (q^2 - v^2)^2 - Fq(e^{iEt} + e^{-iEt}) \right), \quad (86)$$

where F is a small coupling constant. One could also introduce fermionic excitations (instead of the external field), without however changing the results⁸⁵ essentially.

Let us ask now the following question: what is the probability of the transition from the ground state in the left well to a highly excited state in the right one? To lowest order in the external field, the transition probability from the ground state to the N -th state is given by the well known Fermi's Golden Rule:

$$W = 2\pi\delta(E + E_0 - E_N)|\langle N|q|0\rangle|^2 F^2 \quad (87)$$

where

$$\langle N|q|0\rangle \equiv \int dq \Psi_N^*(q) q \Psi_0(q). \quad (88)$$

The problem is then reduced to the calculation of the matrix element of the position operator between two states with a large difference in energy.

Two approaches have been taken to estimate this matrix element: one uses the semi-classical (W.K.B.) method (which will be followed below) and the other (taken by Bachas) a more formal one.

As is well known, the nonperturbed energy eigenstates of the double well potential are symmetric or antisymmetric under parity. For a level lying much below the height of the central barrier, one can define states localized in the left or right well:

$$|N_L\rangle \equiv \frac{1}{\sqrt{2}}(|N_s\rangle + |N_a\rangle) \quad (89)$$

$$|N_R\rangle \equiv \frac{1}{\sqrt{2}}(|N_s\rangle - |N_a\rangle). \quad (90)$$

It is clear however that when the energy is higher than the top of the barrier these states are no longer localized in the left or right well. In the Weinberg Salam theory in four dimensions, the states corresponding to different Chen-Simons numbers are always well defined (for instance they can be distinguished by the number of lefthanded fermions present). Thus the analogy between the real problem and the quantum mechanical toy model is not self evident for energies above the barrier. But the matrix element is exponentially small in such a situation (see below) so that the precise definition of the final state is probably not crucial.

The matrix element is given, to exponential accuracy, by⁸⁶

$$\langle N_R|q|0_L\rangle = e^{(-S_1+S_2-S_3)}, \quad (91)$$

where

$$S_1 \equiv \frac{1}{2g^2} \int_{-\infty}^{-v} dq (q^2 - v^2), \quad (92)$$

$$S_2 \equiv \frac{1}{2g^2} \int_{-\infty}^{q_D} dq \sqrt{(q^2 - v^2)^2 - \epsilon} \quad (93)$$

$$S_3 \equiv \frac{1}{2g^2} \int_{q_E}^{q_F} dq \sqrt{(q^2 - v^2)^2 - \epsilon}, \quad (94)$$

where $\epsilon \equiv 8Eg^2$. The terms in the exponential are shortened action with energy 0 or E , corresponding to the (complex time) trajectory shown in Fig. 8. To our

purpose the only important contributions come from the parts of the trajectory corresponding to Euclidean time evolution (AB , CD and EF in Fig. 8), because the paths in Minkowski time yield only a phase factor. Note that when $E \rightarrow 0$ the contributions from paths AB and CD tend to cancel each other while the path EF gives the usual tunnelling factor at zero energy.

As the energy increases, it is clear that S_3 diminishes but $S_1 - S_2$ grows with energy. The problem is which tendency dominates²⁷.

The answer has been found for very high and very low energies. Furthermore, an argument has been given for the "half-suppression" i.e, the transition probability at its maximum (which occurs at $E = v^4/8g^2$) being the square root of that at $E = 0$.^{87,94}

The transition probability W is proportional to the square of the matrix element $\langle N|q|0\rangle$. It can be calculated without an explicit knowledge of the trajectory at very high or very low energies. Indeed, first consider the quantity

$$g^2 \frac{d \log W}{d\epsilon} = -g^2 \frac{dS_1}{d\epsilon} + g^2 \frac{dS_2}{d\epsilon} + g^2 \frac{dS_3}{d\epsilon}. \quad (95)$$

The right hand side can be simplified: if $\epsilon < 1$ it is

$$g^2 \frac{d \log W}{d\epsilon} = \frac{1}{2} \frac{K(k)}{\sqrt{1+\sqrt{\epsilon}}} \quad (96)$$

while if $\epsilon > 1$ it is equal to

$$g^2 \frac{d \log W}{d\epsilon} = \frac{-K(l)}{2\sqrt{2\sqrt{\epsilon}}}, \quad (97)$$

where $K(x)$ is the complete elliptic integral of the first kind⁹⁷, $k = \sqrt{\frac{1-\sqrt{\epsilon}}{1+\sqrt{\epsilon}}}$ and $l = \sqrt{\frac{\sqrt{\epsilon}-1}{2\sqrt{\epsilon}}}$.

After integration in ϵ one finds the following results:

$$g^2 \log W = -\frac{4}{3} + \frac{\epsilon}{8} \left(\log \frac{64}{\epsilon} + 1 \right) + O(\epsilon^2) \quad (98)$$

for $\epsilon \ll 1$;

$$g^2 \log W = -\frac{\epsilon^{3/4}}{6} B\left(\frac{1}{4}, \frac{1}{2}\right) + O(\epsilon^{1/4}), \quad (99)$$

for $\epsilon \gg 1$, where $B(x, y)$ is the Euler Beta function⁹⁷.

These results show that the transition probability is exponentially suppressed both at low energies (where W grows with energy) and at high energies (where W is exponentially damped with energy).

To demonstrate the square-root suppression of the transition probability at its maximum, one compares the (imaginary) time along the paths CD and EF . According to the classical mechanics

$$g^2 \frac{dS_I}{d\epsilon} = T_I(\epsilon), \quad (100)$$

where $T_I(\epsilon)$ is the time along the I -th path with energy E . The time interval $T(\gamma)$ of the piece of the contour γ is thus given by

$$T(\gamma) = \int_{\gamma} \frac{dq}{\sqrt{U(q) - \epsilon}}. \quad (101)$$

Let us consider the difference of time intervals along the contours γ_1 and γ_2 (Fig. 9), in the complex q plane. As long as the potential has no singularities for finite values of q , the integrand of Eq.(101) has only cuts at the classical turning points. The cut being the square-root type, the value of the integrand at both sides of the cuts differs only in sign. Using Cauchy's Theorem one gets then

$$T_1 + T_2 = 2T_3 \quad (102)$$

or

$$-\frac{dS_3}{d\epsilon} = 2\frac{dS_1}{d\epsilon} - 2\frac{dS_2}{d\epsilon} \quad (103)$$

and after integration in ϵ , this yields

$$S_3(\epsilon) = S_3(0) - 2S_1(\epsilon) + 2S_2(\epsilon). \quad (104)$$

At $\epsilon = 1$, S_3 vanishes, thus

$$S_1(1) - S_2(1) = \frac{1}{2}S_3(0). \quad (105)$$

This result connects the overlap integral (the left hand side of the equation) to the tunnelling factor at zero energy, as anticipated. This is a very general result^{87,94}, independent of the explicit form of the double well used above.

Essentially the same result has been obtained by Bachas⁹⁵, who was able to give a *rigorous* proof of exponential suppression of the induced high-frequency transition amplitudes in an anharmonic potential (one or two wells), based on exact recursion relations between matrix elements of powers of the position operator. This demonstration settles the issue for the quantum mechanical problem, but, in spite of optimistic expectations of the author, no generalization to field theory is known, up to now. Finally we find it interesting that the high energy behaviour of the "Holy Grail function" (for the case of the double well), Eq.(99), found by the WKB method, is consistent with the exact bound found by Bachas, apart from a logarithmic term which is probably beyond the precision of the semi-classical approximation.

Landau's result, Eq.(91), has been earlier generalized to field theory by Iordan-skii and Pitaevskii⁹³. The asymptotic behavior of the imaginary part of the Fourier transform of a two point Green function is given by:

$$Im G(\mathbf{k}, \omega) \sim e^{2(-\Delta S_I(0,0) + \Delta S_{II}(\omega, \mathbf{k}))} \quad (106)$$

The two terms in the exponent correspond to the value of the action along the trajectories at energy $E = 0$ and $E = \omega$. This formula is valid if ω and k are large quantities, and is similar to the quantum mechanical one.

The method of Iordanskii and Pitaevskii was recently applied to non Abelian Gauge theories⁹⁴.

By analogy to the quantum mechanical example one must find singular solutions to the classical Yang-Mills-Higgs equations of motion at energy 0 and E . Diakonov and Petrov look for an $O(3)$ symmetric solution in the pure gauge theory. At 0 energy these are the instanton and anti-instanton with the size ρ^2 changed to $-\rho^2$, while it is more difficult to find a solution if the energy is non zero⁹². In the limits of very high or very low energies an approximate solution can however be obtained analytically. In the low energy regime, the two contributions to the reduced action corresponding to the path ABCD and EF in fig. 8 sum up and give the usual leading result Eq.(18).

In the high energy limit, the cross section for isotropic multiparticle production is found to decrease exponentially with the energy as

$$\sigma(E) \sim e^{-\text{const.}(\alpha\rho E)^{3/5}/\alpha}. \quad (107)$$

(valid for fixed ρ such that $\alpha\rho E \gg 1$.) This result can be interpreted⁹⁴ in terms of an effective strong repulsive interaction between the "instanton" and "anti-instanton" at small separation. Such a repulsion prevents the instanton pair from collapsing, in contrast to what happens in the case of the valley (Section 5). (See Klinkhamer⁹⁸ for a related discussion.)

The result should be correct also in the electroweak model, because both at very high energies and low energies one can neglect the effects of Higgs boson, while at the sphaleron energy one should consider the complete Yang-Mills-Higgs coupled equations.

A naive extrapolation of the results at intermediate energies shows that the maximum of the cross section is achieved near the sphaleron energy where it is close to the square root of its value at zero energy. One cannot draw any conclusive answer because it is not clear whether the particular classical solution used maximizes the multiparticle production cross section.

Finally the work of Cornwall and Tiktopoulos^{89,90} should be mentioned. These authors use a functional Schrödinger equation approach to study $SU(2)$ gauge theory, exploiting the analogy with the quantum mechanical double-well problem. First it is argued that the matrix element,

$$\langle NE|\phi(0)|0\rangle = \int d\phi \psi_E^*\{\phi\}\phi\psi_0\{\phi\} \quad (108)$$

(where the final state with energy E consists of N particles, ϕ is the field operator), is bounded from above simply by

$$\psi_0\{\phi_{NE}\}. \quad (109)$$

ϕ_{NE} is the appropriate field configuration with quantum numbers E , N . Secondly,

the vacuum wave functional for the $SU(2)$ gauge theory is known⁹¹ ($\phi = \mathbf{A}$):

$$\psi_0\{\mathbf{A}\} = \mathcal{N} \sum_{J=-\infty}^{\infty} \exp \left[-\frac{2\pi}{\alpha} |W(\mathbf{A}) - J| \right], \quad (110)$$

where $W(\mathbf{A}) = \mathcal{N}_{CS}$ is the Chern Simons functional (see Eq.(178)). By approximating the final state configuration \mathbf{A} by the sphaleron for which $W(\mathbf{A}) = 1/2$, and by keeping the $J = 1$ term only, these authors get the half (or the square root) suppression of $\Delta(B + L) \neq 0$ cross sections.

4. Unitarity Bounds

Various arguments based on unitarity, which all lead to the "half-suppression" of the $\Delta(B + L) \neq 0$ cross sections, are reviewed in this section.

4.1. Multi-instanton unitarization, "half-suppression", "premature unitarization" and all that

The result of the original calculation of Ringwald²⁰ and Espinosa²¹ violates unitarity if extrapolated to high energies, for reasons similar to those for which the Fermi theory of weak interactions violated unitarity. The effective interaction vertex is given by a local, non-renormalizable form^{99,100,59}:

$$L_{eff} = \int dx \int \frac{d\rho}{\rho^5} \int du d(\rho) e^{-2\pi^2 \rho^2 \bar{\phi}(x) \phi(x)} \cdot \left(e^{\frac{2\pi^2 i}{g} \rho^2 \text{Tr}\{\sigma_\mu \bar{\sigma}_\nu G_{\mu\nu}(x)\}} + e^{\frac{2\pi^2 i}{g} \rho^2 \text{Tr}\{U \bar{\sigma}_\mu \sigma_\nu \bar{U} G_{\mu\nu}(x)\}} \right) \quad (111)$$

where $U \equiv U_\mu \sigma_\mu$, $\bar{U} \equiv U_\mu \bar{\sigma}_\mu$ are the matrices representing a global orientation of the instanton in the isospin space, and the instanton density $d(\rho)$ contains $\exp(-S^{inst})$ and the functional determinant.

The use of Eq.(111) in the lowest order to compute the S matrix elements will necessarily lead to violation of unitarity. Contributions which are higher orders in Eq.(111) - multi-instanton contributions - have to be taken into account in order to restore unitarity¹⁰³⁻¹¹¹.

Zakharov¹⁰³ (also Aoyama and Kikuchi¹⁰⁴, Veneziano¹⁰⁶) pointed out that such multi-instanton corrections can change qualitatively the result found in the original one instanton calculation. A rough estimate of the multi-instanton contribution to the $2 \rightarrow N$ amplitude, iterated in the s-channel (see Fig. 10) and summed over the number of instantons, would give an answer,

$$\begin{aligned} A_{2 \rightarrow N} &\sim \sum_{k=0}^{\infty} A_{2 \rightarrow N}^{tree} \cdot (i A_{N \rightarrow N}^{tree})^{2k} \\ &\sim A_{2 \rightarrow N}^{tree} / \{1 + (A_{N \rightarrow N}^{tree})^2\}, \end{aligned} \quad (112)$$

where a simple factorized form of k - instanton contribution is assumed, the superscript "tree" indicates the single instanton (or anti-instanton) approximation, and

N stands for the dominant, multiparticle intermediate states (it includes symbolically also the summation over them). Although such a simple, factorized structure of Eq.(112) is by no means obvious, a more careful estimation of multi-instanton contribution will be given in the next subsection which will reproduce essentially all the results following from Eq.(112).

The single instanton amplitudes can be taken, for the purpose of resummation, to be of order,

$$A_{2 \rightarrow N}^{tree} \sim e^{-2\pi/\alpha} \cdot K^{1/2}, \quad (113)$$

and

$$A_{N \rightarrow N}^{tree} \sim e^{-2\pi/\alpha} \cdot K, \quad (114)$$

where K is the factor due to the sum over soft gauge bosons produced by instantons. At low energies K is known to grow exponentially as,

$$K \sim e^{+c \frac{2\pi}{\alpha} (\frac{E}{E_{sp}})^{4/3}}; \quad c = O(1) > 0. \quad (115)$$

The difference between Eq.(113) and Eq.(114) reflects the fact that in the latter there is a sum over both the initial and final (multiparticle) states while in the former the summation is only over the final states.

It follows from Eq.(112) that the unitary amplitude is limited by

$$A_{2 \rightarrow N} = \frac{BK^{1/2}}{1 + B^2 K^2} \leq \text{const. } B^{1/2}, \quad B \equiv e^{-2\pi/\alpha}, \quad (116)$$

hence

$$\sigma_{\Delta(B+L) \neq 0} \sim |A_{2 \rightarrow N}|^2 \leq \exp - \frac{2\pi}{\alpha}, \quad (117)$$

the well-known "half-suppression" result. The terminology is due to the fact that at the tree level $\sigma_{\Delta(B+L) \neq 0} \sim \exp - \frac{4\pi}{\alpha}$.

Note that this bound (Eq.(116), Eq.(117)) is independent of the way K depends on the energy, i.e., does not depend on the low-energy approximation, Eq.(115). It is thus probably of a broader validity than it might first appear to be. On the other hand, there is no guarantee that the upper bound is actually reached: the correct statement is that $\sigma_{\Delta(B+L) \neq 0}$ is suppressed *at least* by the half (or more properly, the square root) of the standard 't Hooft factor.

If one does use the leading semi-classical result, Eq.(115), a somewhat stronger result follows. Namely, the full amplitude would be dominated by the multiinstanton terms in Eq.(112), as soon as the energy reaches a critical value such that $A_{N \rightarrow N}^{tree} \simeq 1$ (it turns out to correspond to the value $E_{max} \simeq 0.95 E_{sp}$, in the leading semi-classical approximation: see Eq.(139)). But at this energy the tree amplitude $A_{2 \rightarrow N}^{tree}$ is still exponentially small!

It would appear then that the multi-instanton corrections overwhelm the single instanton contribution at an energy such that the latter is still exponentially small, invalidating any argument based on single instanton calculation extrapolated to higher energies. This scenario was termed "premature unitarization" by Maggiore and Shifman¹¹⁰.

At present^{112–114}, there is no rigorous proof of premature unitarization, as formulated above. The estimate Eq.(115), based on the leading semi-classical approximation, may receive important modifications when higher order corrections are taken into account. However, the alternative possibility - that the multi-instanton contributions never overcome the single instanton term - means that the K factor appearing in Eq.(112), Eq.(116) is never able to compensate the original t'Hooft suppression factor. In other words, the single instanton calculation may be saved, but it must stay exponentially small, after all!

This discussion illustrates the essence of the arguments based on unitarity. Independent of the details (for instance, whether or not premature unitarization holds true), multi-instanton unitarization implies the exponential (at least "half-") suppression of $\sigma_{\Delta(B+L) \neq 0}$.

A similar conclusion, in fact, follows also from a purely Minkowski picture of our processes: the sphaleron resonance formation and its decay^{5,18,19}.

Consider the Breit-Wigner formula for the $2 \rightarrow N$ amplitude

$$A_{2 \rightarrow N} \simeq \frac{\Gamma_2^{1/2} \Gamma_N^{1/2}}{(E - E_{sp}) + i\Gamma_{tot}}, \quad (118)$$

where $\Gamma_{tot} \simeq \Gamma_N$ is the total decay width, Γ_2 is the partial width to the two particle state. Eq.(118) satisfies unitarity, $\text{Im} A_{2 \rightarrow 2} = \sum_N |A_{2 \rightarrow N}|^2$. Now, the sphaleron is a classical field with spatial dimension $\sim 1/gv$, hence the decay particles will have momenta of the order of $\sim gv$. It follows that the average number of the secondaries is $\bar{N} \sim E_{sp}/gv \sim 1/\alpha$ (recall $E_{sp} \sim v/g$.) If one assumes a Poisson distribution for the number of the secondaries - as is appropriate for a coherent state - then one ends up with the estimate

$$|A_{2 \rightarrow N}|^2 \leq \Gamma_2/\Gamma_{tot} \sim e^{-\bar{N}} \sim e^{-\text{const.}/\alpha}. \quad (119)$$

Thus the cross section for the sphaleron formation is small because the latter is coupled weakly to the two-particle initial state.

The fact that we find from the sphaleron picture an estimate similar to the one obtained in the multi-instanton picture, is not really surprising. The sphaleron is an unstable state staying on the top of the barrier separating two adjacent vacua of the SU(2) gauge theory⁵, decaying with equal probabilities to $\Delta(B+L) \neq 0$ and $\Delta(B+L) = 0$ final states. In the context of Euclidean, instanton description of $\Delta(B+L) \neq 0$ transition, the sphaleron should correspond to multi (infinite) instanton configurations.

Finally, it is possible to understand both types of arguments based only on a very general feature of unitarity and the particular aspect of involved dynamics - dominantly multi-particle production.

Consider¹⁰⁸ the s-channel unitarity equation for the forward elastic amplitude $f_1 f_2 \rightarrow f_1 f_2$ ($A_{el} \equiv A_{2 \rightarrow 2}$),

$$\text{Im} A_{el} = \sum_N |A_{2 \rightarrow N}|^2$$

$$\begin{aligned}
&= |A_{el}|^2 + \sum_{N \neq 2} |A_{2 \rightarrow N}|^2 \\
&\propto \sigma_{el} + \sigma_{inel},
\end{aligned} \tag{120}$$

where we assumed a simple form of unitarity appropriate for the S-wave¹⁰², that is adequate for processes described by a single instanton or sphaleron, and neglected all complications arising from spin. As is well known, it leads to the upper limit for an S-wave amplitude,

$$|A_{el}| \leq 1. \tag{121}$$

The question is whether such a limit can be actually saturated in the case of instanton-induced multi W boson productions.

It should be recalled that in the case of high energy hadron hadron scattering (due to strong interactions), more and more partial waves come into play as the energy increases, and this leads to a much less stringent unitarity limit, the so-called Froissart bound¹⁰¹ for the total cross section,

$$\sigma_{tot} \propto \log^2 s. \tag{122}$$

(which amounts to $|A_{el}(s, 0)| \leq s \log^2 s$.)

Whether the limit (121) can be reached, depends on the dynamics, even if the reaction proceeds indeed through the S-wave only. Let us define the inelasticity r ,

$$r \equiv \sigma_{inel}/\sigma_{el} = \sum_{N \neq 2} |A_{2 \rightarrow N}|^2 / |A_{el}|^2 \tag{123}$$

Eq.(120) and Eq.(123) together imply a more stringent bound,

$$\sigma_{\Delta(B+L) \neq 0} \leq \sigma_{tot} \simeq \frac{1}{s} \text{Im} A_{el} \leq \frac{1}{s} \frac{1}{1+r}. \tag{124}$$

We see that whatever mechanism (in pure S-wave) gives $r \gg 1$ would lead to $\sigma_{\Delta(B+L) \neq 0}$ much smaller than the geometrical value ($\sim \frac{1}{s}$).

In a more careful treatment one should distinguish the baryon number violating from baryon number conserving intermediate states in Eq.(120). The argument essentially goes through without modification, though, the reason being that the particular mechanism under study - multi-instanton sum - will give the same contribution to $\Delta(B+L) \neq 0$ and $\Delta(B+L) = 0$ processes, at its maximum (see the next subsection).

The s-channel iteration of instanton anti-instanton chain is a crude approximation which ensures the s-channel unitarity, Eq.(120). A four point amplitude must satisfy the t-channel (as well as the u-channel) unitarity¹⁰² also. That requires at least the instanton chain iterated also in the t- or u-channel. Such contributions can modify the S-wave dominance of the amplitude: cannot the electroweak high energy scattering then become of multi-peripheral type at high energies, with a much larger cross section, and become similar to the "soft" hadronic processes?

Such a possibility cannot in principle be excluded. Nevertheless, there is a difference here as compared to the multi-instanton contribution iterated in the direct (s-) channel. The large energy (large s) carried by the initial particles is converted into the energy of multiple gauge or Higgs bosons. The effect of the final state summation is to enhance dramatically the leading instanton amplitude^{20,21}. The effect is so large as to lead to the violation of s-channel unitarity: one is *forced* to take into account higher order (multi-instanton) contributions, iterated in the s-channel. On the contrary, no reasons are known to suspect that multi-instantons iterated in the t-channel (for instance) are of any particular importance. Multi-instanton chains iterated in the t or u channel presumably remain negligibly small.

4.2. Resummation of multi-instanton contributions

These arguments above, leading to the "half-suppression", based on a very rough estimate, Eq.(112), will be corroborated below by actually resumming the multi-instanton contributions. The treatment closely follows that by Musso and one of the authors¹⁰⁸.

The starting point will be the one-instanton R -term calculation of the $\Delta(B + L) \neq 0$ cross section, or the imaginary part of the forward elastic amplitude, reviewed in Section (1.3). To generalize the formula to multi-instanton cases one needs the knowledge of the *amplitude itself* rather than its imaginary part, since the former (more precisely the S-matrix element) can be iterated in the sense of Feynman diagrams.

To compute the amplitude, we recall that if the instantons and anti-instantons are far apart, the single instanton S-matrix elements can be treated as an effective Lagrangian, L_{inst} . The result Eq.(28), Eq.(29), Eq.(30) just corresponds to the lowest contribution in L_{inst} to the amplitude of W-boson production, squared and summed over the final states. Thus to get the amplitude, one replaces the "cut propagator"

$$\pi \int \frac{d\mathbf{k}}{\omega_k} = 2\pi \int d^4k \theta(k^0) \delta(k^4) \quad (125)$$

appearing in Eq.(29) by the uncut propagator

$$\int \frac{d^4k i}{k^2 + i\epsilon}. \quad (126)$$

One finds thus,

$$\langle \mathbf{p}, -\mathbf{p} | A | \mathbf{p}, -\mathbf{p} \rangle = i e^{-16\pi^2/g^2} \int dt d\mathbf{x} d\rho d\rho' C(\rho, \xi, \rho', \xi') e^{W'}. \quad (127)$$

The exponent W' is given by

$$W' = -iEt - \pi^2 v^2 (\rho^2 + \rho'^2) + \frac{\pi}{g^2} \rho^2 \rho'^2 \int \frac{d^4k i}{k^2 + i\epsilon} e^{ikx} (3k_0^2 + \mathbf{k}^2). \quad (128)$$

Now, since

$$\int \frac{d^4k i}{k^2 + i\epsilon} e^{ikx} = -\frac{4\pi^2}{x^2 - i\epsilon}, \quad (129)$$

the integral in k in W' is equal to

$$\frac{32\pi^2(3x_0^2 + \mathbf{x}^2)}{(x^2 - i\epsilon)^3}. \quad (130)$$

Computing the \mathbf{x} integration by the saddle point method, one finds at the saddle point ($\mathbf{x} = 0$),

$$W' = -iEt - \pi^2 v^2(\rho^2 + \rho'^2) + \frac{96\pi^2 \rho^2 \rho'^2}{g^2 t^4} = W : \quad (131)$$

namely, in the leading order we find the same exponent for the forward amplitude as for its imaginary part. This is consistent with the fact that the leading $i - a$ contribution to the forward amplitude is purely imaginary. Computing the resulting integrals over t and ρ, ρ' by the saddle point method as before, one finds thus

$$\langle \mathbf{p}, -\mathbf{p} | A | \mathbf{p}, -\mathbf{p} \rangle = i \exp\left(-\frac{16\pi^2}{g^2} + 3\left(\frac{3E^4}{8\pi^2 g^2 v^4}\right)^{1/3}\right), \quad (132)$$

in accordance with Eq.(18).

The contribution from n pairs of $i - a$ (Fig. 10) to the forward amplitude can be written down by generalizing Eq.(127), Eq.(128). If we neglect various pre-exponential factors, it reads,

$$\langle \mathbf{p}, -\mathbf{p} | A^{(2n)} | \mathbf{p}, -\mathbf{p} \rangle = i^{2n-1} e^{-16\pi^2 n/g^2} \int \dots \int \prod_{i=1}^{2n-1} dt_i \prod_{i=1}^{2n} d\rho_i \exp W. \quad (133)$$

$$W = -iE \sum_{i=1}^{2n-1} t_i - \pi^2 v^2 \sum_{i=1}^{2n} \rho_i^2 + \frac{96\pi^2}{g^2} \sum_{i=1}^{2n-1} \rho_i^2 \rho_{i+1}^2 / t_i^4, \quad (134)$$

In arriving at Eq.(134) the integrations over the relative instanton orientations have been done by the saddle point approximation which effectively set all instantons and anti-instantons alligned in the $SU(2)$ space. The integration over the relative instanton (space) positions yields

$$\mathbf{x}_1 = \mathbf{x}_2 = \dots = \mathbf{x}_{2n-1} = 0. \quad (135)$$

The remaining integrations over the relative instanton time coordinates t_i , and their sizes ρ_i , can also be done easily. See Appendix A.

The resulting $2n$ instanton contribution to the forward $ff \rightarrow ff$ amplitude is now (in terms of the dimensionless variable $x \equiv E/E_{sp}$)

$$A^{(2n)} \propto i^{2n-1} H^{2n-1} e^{\frac{-16\pi^2}{g^2}((1-1.0817x^{4/3})n+0.7672x^{4/3})}, \quad (136)$$

where the real factor H contains the power contribution from the fermion zero modes in the alternative $i - a$ "bonds" as well as the preexponential factors arising from the gaussian integrations.

Note that the phase factor i^{2n-1} in $A^{(2n)}$ originates from the continuation from $2n$ $i - a$ centers in the Euclidean space to Minkowski spacetime positions. (One missing i is due to the standard definition of the amplitude.) All other Gaussian integrations over t_i, s and ρ_i, s are real. As a result the sign alternates in the sum over n (see Eq.(112)), and gives

$$A \equiv \sum_{n=1}^{\infty} A^{(2n)} \simeq i \frac{H e^{\frac{-16\pi^2}{g^2}[1-1.0817x^{4/3}]} }{1 + H^2 e^{\frac{-16\pi^2}{g^2}[1-1.0817x^{4/3}]} } e^{-\frac{16\pi^2}{g^2}[0.7672x^{4/3}]} . \quad (137)$$

This concludes the calculation of the forward elastic amplitude in the leading semi-classical n -instanton approximation, resummed over n .

There are several noteworthy features in Eq.(137).

(i) As expected, the resummed amplitude behaves in a way qualitatively different from the single instanton contribution ($n = 1$ term only). The resummed amplitude, which is unitary, turns out to be exponentially suppressed at all energies. It reaches the maximum

$$|A|_{max} \simeq \exp -\frac{2\pi}{\alpha}(1.333) \quad (138)$$

at

$$E_{max} \simeq 0.95 E_{sp}. \quad (139)$$

where $1 - 1.0817x^{4/3} \simeq 0$. At higher energies it is exponentially damped. The suppression Eq.(139) is somewhat stronger than the naive "half suppression" factor $\exp(-2\pi/\alpha)$, in agreement with our general conclusion.

(ii) That the resummed amplitude $A \equiv \sum_{n=1}^{\infty} A^{(2n)}$ satisfies the s-channel unitarity,

$$(A_{2 \rightarrow 2} - A_{2 \rightarrow 2}^*)/2i = \sum_N A_{2 \rightarrow N} \times A_{2 \rightarrow N}^*, \quad (140)$$

can be seen as follows. Consider $A_{2 \rightarrow 2}$ and $A_{2 \rightarrow N}$, both computed in the leading $2n$ -instanton approximation, summed over n . Compare the two sides of the unitarity equation, namely $Im A_{2 \rightarrow 2}$ and $\sum_N |A_{2 \rightarrow N}|^2$. The latter can be written as

$$\sum_N \sum_{l,m=1}^{\infty} A_{2 \rightarrow N}^{(l)} (A_{2 \rightarrow N}^{(m)})^* = \sum_N \sum_{n=1}^{\infty} \sum_{l=1}^{2n-1} A_{2 \rightarrow N}^{(2n-l)} (A_{2 \rightarrow N}^{(l)})^*. \quad (141)$$

Recall that the sum over the intermediate states N (corresponding to cut propagators) gives rise to the same exponent as the sum over virtual intermediate states (with uncut propagators), as noted before (after Eq.(131)). Taking into account the appropriate phase factor the (n, l) term of Eq.(141) is

$$\sum_N A_{2 \rightarrow N}^{(2n-l)} (A_{2 \rightarrow N}^{(l)})^* = i^{2n-l-1} (-i)^{l-1} |A_{2 \rightarrow 2}^{(2n)}|. \quad (142)$$

These terms are seen to be in one-to-one correspondence with various cuts of multi-instanton chain for $A_{2 \rightarrow 2}^{(2n)}$. Indeed, the identity,

$$\sum_{l=1}^{2n-1} i^{2n-l-1} (-i)^{l-1} = (-)^{n-1} = (i^{2n-1} - (-i)^{2n-1})/2i \quad (143)$$

shows that Eq.(140) is satisfied in this approximation.

(iii) Note that both types of cuts corresponding to $\Delta(B+L) \neq 0$ and $\Delta(B+L) = 0$ intermediate states appear in the unitarity equation. However, it is not difficult to separate the two types of contributions in Eq.(137), once one identifies the contributions of various cuts, Eq.(143). One finds

$$\text{Im } A = \text{Disc}_{\Delta(B+L) \neq 0} A + \text{Disc}_{\Delta(B+L) = 0} A. \quad (144)$$

$$\text{Disc}_{\Delta(B+L) \neq 0} A = \frac{H e^{\frac{-16\pi^2}{g^2}[1-1.0817x^{4/3}]}}{(1 + H^2 e^{\frac{-16\pi^2}{g^2}[1-1.0817x^{4/3}]})^2} e^{-\frac{16\pi^2}{g^2}[0.7672x^{4/3}]}, \quad (145)$$

$$\text{Disc}_{\Delta(B+L) = 0} A = \frac{H^3 e^{\frac{-32\pi^2}{g^2}[1-1.0817x^{4/3}]}}{(1 + H^2 e^{\frac{-16\pi^2}{g^2}[1-1.0817x^{4/3}]})^2} e^{-\frac{16\pi^2}{g^2}[0.7672x^{4/3}]}, \quad (146)$$

and

$$\sigma_{\Delta(B+L) \neq 0} \propto \left(\frac{1}{s}\right) \text{Disc}_{\Delta(B+L) \neq 0} A. \quad (147)$$

Clearly all the point made above in (i) applies to $\sigma_{\Delta(B+L) \neq 0}$ which is the quantity one is really interested in.

Finally let us note that at the energy Eq.(139) at which $\sigma_{\Delta(B+L) \neq 0}$ takes the maximum value (and where $1 - 1.0817x^{4/3} \simeq 0$), the baryon number violating and conserving cross sections are the same:

$$\sigma_{\Delta(B+L) \neq 0} \simeq \sigma_{\Delta(B+L) = 0} \quad (148)$$

to exponential accuracy, in accordance with the sphaleron resonance formation picture.

(iv) Instead of $2 \rightarrow N$ amplitude considered above, one can study $N \rightarrow N'$, with the coherent states method of Khlebnikov et al³⁵. Consider in particular the forward amplitude,

$$ff + N \text{ gauge bosons} \longrightarrow ff + N \text{ gauge bosons}, \quad (149)$$

in the multi-instanton approximation, summed over N . The R-term technique for doing the sum over the number of particles in the intermediate states used above can be easily extended to the initial state (see also Section 2.2).

One finds that

$$|A_{ff+N \rightarrow ff+N}|_{N=N_{max}} \simeq \left| \sum_{\text{all } N} A_{ff+N \rightarrow ff+N} \right| \quad (150)$$

for some N_{max} of order of $O((\frac{1}{\alpha})(\frac{E}{E_{sp}})^{4/3})$ by using the well known relation between a coherent state and a state with a given number of particles. Also, the right hand side of this equation can be computed just as in the case of the forward $2 \rightarrow 2$ amplitude, Eq.(137). The crucial difference is however that now no "end point effect" is present and as a result one gets¹⁰⁸

$$A_{ff+N \text{ gauge bosons} \rightarrow ff+N \text{ gauge bosons}}|_{N=N_{max}} = i \frac{H e^{\frac{-16\pi^2}{g^2}[1-1.0817x^{4/3}]}}{1 + H^2 e^{\frac{-16\pi^2}{g^2}[1-1.0817x^{4/3}]}}. \quad (151)$$

a result similar to Eq.(137) but without the last exponential factor. Thus the many-to-many amplitude would reach the unitarity limit, $\exp -\frac{4\pi}{\alpha} \cdot 0$, in contrast to the $2 \rightarrow$ all amplitude.

4.3. High temperature or high density transition as compared to processes at high energies

As is well known¹², high-temperature $\Delta(B+L) \neq 0$ transitions occurs classically (and without barrier penetration factor) in the standard model, if the temperature is above the energy barrier, that is if

$$T \geq E_{sp} \quad (152)$$

Analogously, the ground state with finite fermion density will decay without any suppression if the Fermi sea level is sufficiently high, i.e., of the order of the sphaleron mass¹³.

That a similar disappearance of the tunnelling factor might occur also in *high-energy* scattering, was the original motivation for the earliest works^{20,21}. However, there is a clear physics difference here. A characteristic feature of the scattering processes is that the high energy of the initial channel is carried by just two energetic particles. In the high temperature (or high density) transitions, on the contrary, it is expected that the relevant initial states, at a given (high) energy and with a given set of quantum numbers, are mainly multiparticle states. For there are many more such states as compared to two particle states, hence they will dominate the process for purely statistical reasons. Thus although many-to-many amplitudes might well become unsuppressed at the sphaleron energy (see Eq.(151)), the two-to-many amplitude will remain small. A suggestion put forward by Diakonov and Petrov^{15,16}, that the decay rate of the state with a finite density (with the Fermi level μ) is proportional to $\Delta(B+L) \neq 0$ cross section (with $E \sim \mu$), seems to be invalidated, once this difference in the initial states are taken into account¹⁷.

5. Fermions in the valley

In this section the whole problem will be analyzed from a somewhat different point of view. The behavior of the chiral fermions in the valley background is studied, and the $\Delta(B+L) \neq 0$ cross section is analysed by using the optical theorem.

5.1. Unitarity puzzle and Spectrum of the Dirac operator

As already noted in Sections (1.3) and (4.1), the total $\Delta(B+L) \neq 0$ cross section can be computed via unitarity, i.e., as an appropriate part of the imaginary part of the forward elastic amplitude,

$$1 + 2 \rightarrow 1 + 2. \quad (153)$$

However, a problem concerning unitarity and chiral anomaly arises, which can be formulated as follows¹¹⁵. The optical theorem states that the cross section,

$$1 + 2 \rightarrow X, \quad (154)$$

summed over all possible X , is equal, apart from a kinematical factor, to the imaginary part of the forward elastic amplitude, Eq.(153). Now consider a particular class of processes (154) induced by an $SU(2)$ instanton, with

$$\Delta(B+L) \neq 0; \quad \Delta(B-L) = 0. \quad (155)$$

Sum over the final states satisfying (155) should give *a part* of the full imaginary part of the elastic amplitude:

$$Anom \text{ Im } A_{2 \rightarrow 2} = \sum_{\Delta(B+L) \neq 0} |A_{2 \rightarrow X}|^2, \quad (156)$$

see Eq.(145-146). Now, for an (anti-) instanton background, which is relevant for the calculation of the right hand side of Eq.(156), each right (left) handed fermion field has a zero mode. The standard functional integration over fermions yields a product of these zero modes; by going to momentum space and by applying the LSZ amputation one finds the S-matrix elements consistent with the instanton selection rule Eq.(155).

How to calculate the left hand side of Eq.(156)? Being a part of the elastic amplitude, it must arise from a four point function computed in a background, topologically (globally) equivalent to the trivial, perturbative vacuum. To be equal to the total anomalous ($\Delta(B+L) \neq 0$) cross section, however, such a background must have a nontrivial topological structure, for instance, similar to a widely separated instanton anti-instanton pair. More precisely, the work of Arnold and Mattis⁵² suggests that one should consider something like the valley.

The problem is that no fermion zero modes exist in the valley background¹¹⁵ (see Appendix B for a sketch of the proof). The spectrum of the Dirac operator in the valley background Eq.(34) is proven¹¹⁵ to be the same as that of the free Dirac operator, with the continuum spectrum $(-\infty, \infty)$ and with no normalizable zero or nonzero modes. Thus one wonders how the left hand side of Eq.(156) can be computed, which should somehow be dominated by the standard lefthanded or righthanded zero modes, in order to match the right hand side.

Note that this "unitarity puzzle" by no means depends on the use of the fermions as external particles in an essential manner, although the puzzle looks much neater

with fermions. In the literature, often theories without fermions are considered, on the basis that the main problem lies in topologically nontrivial aspects of *gauge* field dynamics (which is quite true). The use of gauge bosons as external particles however does not eliminate the unitarity puzzle. For instance, the validity of the leading order "semiclassical" approximation (i.e. substituting the instanton solution in the external lines) is not at all obvious for the left hand side of Eq.(156) (especially in the regime of strongly overlapping instantons), if it is more reasonable for the production amplitude appearing on the right hand side.

To solve the "unitarity puzzle", one must study the elastic amplitude starting from the four point function computed in an appropriate background of instanton anti-instanton type (which will be approximated here by the valley), and must show that the imaginary part of the forward elastic amplitude indeed contains an anomalous piece which reproduces the right hand side of Eq.(156). The work reviewed in the next section is a first step towards such a solution.

5.2. Fermion Green function in the valley: widely separated instanton anti-instanton pair

The four point function,

$$\begin{aligned} & \langle T\psi_1(x)\psi_2(u)\bar{\psi}_1(y)\bar{\psi}_2(v) \rangle^{(A_{valley})} \\ &= \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \psi_1(x)\psi_2(u)\bar{\psi}_1(y)\bar{\psi}_2(v) e^{-S} / Z^{(A=0)}; \end{aligned} \quad (157)$$

$$S = \sum_{j=1}^{N_F} \int d^4x i \bar{\psi}_j \bar{D} \psi_j \quad (158)$$

in the fixed background of Eq.(34), has been studied by two of the present authors¹¹⁵. As the functional integral factorizes in flavour the quantity of interest are (suppressing the flavour index),

$$I(x, y) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \psi(x) \bar{\psi}(y) e^{-\int d^4x i \bar{\psi} \bar{D} \psi}, \quad (159)$$

and

$$\mathcal{Z} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\int d^4x i \bar{\psi} \bar{D} \psi} = \det \bar{D}, \quad (160)$$

where it is assumed that $\det \bar{D}$ is suitably regularized.

The key point in this analysis is to introduce complete sets of orthonormal modes $\{\eta_n^{(a)}\}$ and $\{\bar{\zeta}_n^{(i)}\}$, $n = 0, 1, 2, \dots$, for the left-handed and right-handed fermions, respectively. They are eigenstates of $D^{(a)} \bar{D}^{(a)}$ and $\bar{D}^{(i)} D^{(i)}$:

$$\begin{aligned} \bar{D}^{(a)} \eta_m^{(a)} &= \bar{k}_m \bar{\zeta}_m^{(a)} \quad (m = 0, 1, \dots), & D^{(a)} \bar{\zeta}_m^{(a)} &= k_m \eta_m^{(a)} \quad (m = 1, 2, \dots), \\ \bar{D}^{(i)} \eta_m^{(i)} &= l_m \bar{\zeta}_m^{(i)} \quad (m = 1, 2, \dots), & D^{(i)} \bar{\zeta}_m^{(i)} &= \bar{l}_m \eta_m^{(i)} \quad (m = 0, 1, \dots), \end{aligned} \quad (161)$$

where

$$\bar{k}_0 = \bar{l}_0 = 0. \quad (162)$$

The covariant derivatives $D^{(a)}$, $D^{(i)}$ are defined with respect to the antiinstanton and instanton parts of Eq.(34). Accordingly the zero modes are those in the regular gauge (for the lefthanded mode) and in the singular gauge (for the righthanded one), respectively.

The functional integration can then be defined as:

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \equiv \prod_{m,n=0}^{\infty} da_m d\bar{b}_n;$$

$$\psi(x) = \sum_{m=0}^{\infty} a_m \eta_m^{(a)}(x), \quad \bar{\psi}(x) = \sum_{n=0}^{\infty} \bar{b}_n \bar{\zeta}_n^{(i)*}(x). \quad (163)$$

As is clear from the way these equations are written, the system is first put in a large but finite box of linear size L (such that $L \gg R, \rho$) so that all modes are discrete. After the derivation of Eq.(167) below (i.e., after the sum over the complete sets is done), however, L can be sent to infinity without any difficulty.

The two point function $I(x, y)$ can be written as

$$\begin{aligned} I(x, y) &= \det \bar{D} \langle x | \bar{D}^{-1} | y \rangle \\ &= \det \bar{D} \{ \langle x | a, 0 \rangle \langle a, 0 | \bar{D}^{-1} | i, 0 \rangle \langle i, 0 | y \rangle + \sum_{m \neq 0} \langle x | a, m \rangle \langle a, m | \bar{D}^{-1} | i, 0 \rangle \langle i, 0 | y \rangle \\ &\quad + \sum_{n \neq 0} \langle x | a, 0 \rangle \langle a, 0 | \bar{D}^{-1} | i, n \rangle \langle i, n | y \rangle + \sum_{m, n \neq 0} \langle x | a, m \rangle \langle a, m | \bar{D}^{-1} | i, n \rangle \langle i, n | y \rangle \} : \end{aligned} \quad (164)$$

the term proportional to the product of the zero modes has been singled out. We wish to compute $I(x, y)$ at small ρ/R . To do this, first let us write

$$\bar{D} = \begin{pmatrix} d & v_1 & \dots & v_n & \dots \\ w_1 & X_{11} & \dots & X_{1n} & \dots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ w_m & X_{m1} & \dots & X_{mn} & \dots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix} \quad (165)$$

The idea is that the matrix elements involving either of the zero modes, d , v_n , w_m , are all small by some overlap integrals while the matrix elements X_{mn} are large because the wave functions of non zero modes are extended to all over the spacetime.

The inverse matrix \bar{D}^{-1} is given by:

$$\begin{aligned} (\bar{D}^{-1})_{00} &= 1/(d - vX^{-1}w) \\ &= d^{-1} + d^{-2}v_m(X^{-1})_{mn}w_n + \dots; \\ (\bar{D}^{-1})_{mn} &= (X - \frac{1}{d}w \otimes v)^{-1} = X^{-1}(1 - \frac{1}{d}w \otimes vX^{-1})^{-1} \\ &= (X^{-1})_{mn} + d^{-1}(X^{-1})_{ml}w_lv_k(X^{-1})_{kn} + \dots, \\ (\bar{D}^{-1})_{0n} &= -d^{-1}v_l(\bar{D}^{-1})_{ln}, \\ (\bar{D}^{-1})_{m0} &= -(\bar{D}^{-1})_{00}X_{mk}^{-1}w_k, \end{aligned} \quad (166)$$

where X^{-1} is the inverse of the submatrix X in the space orthogonal to the zero modes.

Inserting Eq.(166) into Eq.(164) and after some algebra one finds a remarkably simple (and still exact) expression for $I(x, y)$:

$$I(x, y) = \det X \{ \langle x|a, 0 \rangle - \langle x|X^{-1}\bar{C}|a, 0 \rangle \} \{ \langle i, 0|y \rangle - \langle i, 0|\bar{B}X^{-1}|y \rangle \} + \det \bar{D} \langle x|X^{-1}|y \rangle, \quad (167)$$

where $\bar{C} \equiv C_\mu \bar{\sigma}_\mu$; $\bar{B} \equiv B_\mu \bar{\sigma}_\mu$ in Eq.(167) are defined by:

$$D_\mu^{(valley)} = D_\mu^{(a)} + C_\mu = D_\mu^{(i)} + B_\mu. \quad (168)$$

The function C_μ is the (modified) instanton field while B_μ is the (modified) anti-instanton.

Eq.(167) displays nicely the main features of the two point function in the valley background. The effect due to the zero modes is separated and everything else is expressed by the smoother two point function,

$$S'_{x,y} = \langle x|X^{-1}|y \rangle. \quad (169)$$

$S'_{x,y}$ can be assumed to behave at large x and y (with x_i and x_a fixed) as

$$\begin{aligned} S'_{x,y} &\sim U^\dagger(x) S_F(x, y) U(y), \\ U(x) &= \frac{\bar{\sigma}_\mu(x - x_a)_\mu}{\sqrt{(x - x_a)^2}}, \end{aligned} \quad (170)$$

where S_F is the free Feynman propagator. This behavior is suggested by the fact that the valley field has a pure gauge form at large x ,

$$A_\mu^{(valley)} \sim \frac{i}{g} U^\dagger \partial_\mu U \sim O\left(\frac{1}{x}\right). \quad (171)$$

To proceed further one assumes that

$$\det X / \det \bar{\partial} = \text{const.} \quad (172)$$

as $\rho/R \rightarrow 0$. Next the ratio $\det \bar{D} / \det X$ can be estimated as follows (see (166)):

$$\frac{\det \bar{D}}{\det X} = ((\bar{D}^{-1})_{00})^{-1} \simeq d \simeq \text{const.} \rho^2 / R^3, \quad (173)$$

where use was made of

$$d = \bar{D}_{00} = \langle i, 0|\bar{C}|a, 0 \rangle = \int_z \bar{\zeta}_0^{(i)}(z)^* \bar{C}(z) \eta_0^{(a)}(z) \sim \rho^2 / R^3.$$

Combining Eq.(172) and Eq.(173) gives

$$\frac{\det \bar{D}}{\det \bar{\partial}} \sim \rho^2 / R^3. \quad (174)$$

With Eq.(174) and Eq.(170) in Eq.(167) one can estimate the amplitude and the leading contribution to its anomalous part.

There is a subtlety here. Because the background (in the gauge of Eq.(34)) reduces asymptotically to a pure gauge form, Eq.(171), and does not vanish sufficiently fast, the standard LSZ procedure cannot be used to extract the amplitude. The correct procedure is to go to a more physical gauge

$$\begin{aligned}\tilde{A}_\mu^{(valley)} &= U(A_\mu^{(valley)} + \frac{i}{g}\partial_\mu)U^\dagger, \\ U(x) &= \frac{\bar{\sigma}_\mu(x-x_a)_\mu}{\sqrt{(x-x_a)^2}},\end{aligned}\tag{175}$$

by appropriately transforming the fermion four point function, before the standard LSZ procedure is applied¹¹⁵.

The contribution of the first term of Eq.(167) the elastic amplitude is found to be, to leading order in ρ/R , proportional to

$$\rho^2 \exp(ip \cdot x_a) \exp(-iq \cdot x_i).\tag{176}$$

One finds also a correction proportional to ρ/R times this factor, coming from the terms containing the function B and C in Eq.(167). These terms are precisely what one has been looking for: the part of the elastic amplitude, whose imaginary part is going to match the right hand side of the unitarity equation, Eq.(156).

The second term of Eq.(167), in contrast, mainly gives rise to non-anomalous discontinuity, associated with $\Delta(B+L)=0$ intermediate states.

Thus at least for the valley corresponding to widely separated instanton anti-instanton configuration, the elastic amplitude contains the piece which reproduces the square of the production amplitude computed in the single instanton background.

The *general* recipe for calculating the left hand side of the "anomalous" unitarity relation, Eq.(156), is however not yet known to the best of our knowledge.

We have thus far considered the behavior of fermions in the valley which corresponds to a particular kind of instanton anti-instanton pair configuration, oriented in the maximally attractive direction⁴² and with a particular interaction term⁴⁴. Such a field is supposed to be of particular relevance in the problem of baryon number violation, as reviewed in Section 1.2.

A related problem of the fermion propagation in a background of *simple sum* of widely separated instantons and anti-instantons (in singular gauge), was studied earlier. Such backgrounds might be important in modelling some feature of the physical vacuum of QCD^{4,122,123}. It is Lee and Bardeen¹²⁴ who, in such a context, showed for the first time that the fermion propagator can be in some sense dominated by a term proportional to a product of the standard zero modes (for widely separated instanton-anti-instantons), in spite of the absence of any fermion zero modes in the background considered.

Also, the method used by Lee and Bardeen (they work with the equation satisfied by the fermion Green function, rather than doing the functional integration) is very different with our own; of course, our formula reproduces theirs to the leading order in ρ/R , if one uses the simple sum of instanton and anti-instanton instead of the valley and sets $X^{-1} \simeq S_F + (S_a - S_F) + (S_i - S_F)$.

5.3. Overlapping instanton anti-instanton pair

The valley background used above reduces to the vacuum field at $R = 0$ ⁱ. One wonders whether the transition to a purely perturbative field occurs gradually and only terminates at precisely $R/\rho = 0$, or it takes place abruptly at a finite value of R/ρ , probably of order of unity.

There is a strong indication that the latter possibility is realized. The first indication¹¹⁵ comes from the numerical and analytical study of the integrated topological density,

$$C(x_4) \equiv - \int_{-\infty}^{x_4} \int d^3x \frac{g^2}{16\pi^2} \text{Tr} F_{\mu\nu} \tilde{F}_{\mu\nu} = \mathcal{N}_{CS}(x_4) - \mathcal{N}_{CS}(-\infty), \quad (177)$$

as a function of x_4 for several values of R/ρ , for the valley background of Eq.(34). (See Fig. 11.) $\mathcal{N}_{CS}(x_4)$ is the Chern Simons number

$$\mathcal{N}_{CS}(x_4) \equiv - \int d^3x \frac{g^2}{16\pi^2} \epsilon^{4ijk} \text{Tr} (F_{ij} A_k - \frac{2}{3} A_i A_j A_k). \quad (178)$$

The instanton and anti-instanton are situated at $(\mathbf{0}, R/2)$ and at $(\mathbf{0}, -R/2)$, respectively.

It can be seen from Fig.11 that the topological structure is well separated and localized at the two instanton centers only at relatively large values of R/ρ , $R/\rho \geq 10$: for such R/ρ , $C(x_4)$ reproduces locally the situation of single instanton background (near $x_4 = R/2$) and that of anti-instanton (near $x_4 = -R/2$). Vice versa, for small $R/\rho \leq 1$ the gauge field is seen to collapse to some insignificant fluctuation around zero, not clearly distinguishable from ordinary perturbative ones. (These statements are, admittedly, a little vague and not very precise. See the next subsection for an attempt to make them more precise.)

In Fig. 12 is plotted also the behavior of the maximum of each curve, corresponding to $C(0)$, as a function of R/ρ . The behavior of $C(0)$ is powerlike both at large and small R :

$$\begin{aligned} C(0) &\sim 1 - 12(\rho/R)^4, & R/\rho \gg 1, \\ C(0) &\sim \frac{3}{4}(R/\rho)^2, & R/\rho \ll 1. \end{aligned}$$

(Actually, the exact expression for $C(x_4)$ in terms of x_4 can be found by using conformal transformations¹¹⁶). In particular,

$$C(0) = 3\left(\frac{z-1}{z+1}\right)^2 - 2\left(\frac{z-1}{z+1}\right)^3, \quad (179)$$

ⁱThe triviality of the valley for $R = 0$ is a feature independent of the choice of the weight w ⁴⁴.

where z is defined in (35).)

The behavior of $C(0)$ supports the idea that the transition to a perturbative background is a sharp one. In particular, the quadratic behavior of $C(0)$ in R/ρ found at small R/ρ is nothing but the reflection of the perturbative quadratic fluctuation. This is particularly transparent if one uses the gauge¹¹⁵ in which

$$A_\mu \propto R.$$

Altogether, Fig. 11 and Fig. 12 indicate that the instanton and antiinstanton start to melt at around $R/\rho \simeq 5$ and go through the transition quickly, the center of the transition to a purely perturbative regime being at around $R/\rho = 1$. The known behavior of the valley field action^{43,44}, which sharply drops from near the two instanton value $4\pi/\alpha$ to zero around the transition region, $R/\rho \simeq 1$, (see Fig. 3) is perfectly consistent with this conclusion.

5.4. *Level-crossing in the valley*

These discussions, though convincing intuitively, do not provide a quantitative answer as to the value of R/ρ at which such a transition occurs. The best thing to do would be to analyse the fermion four point function directly around $R/\rho \sim 1$, which however appears to be a difficult task for the time being.

The next-best thing to do is to study the level crossing of the chiral fermion in the background of the valley, and see whether a *qualitative* change occurs at a finite value of R/ρ . The idea is the following.

As is well known, one way to understand the anomalous chiral fermion generation by a topologically non trivial gauge fields,

$$\Delta N_R - \Delta N_L = \int d^4x \frac{1}{16\pi^2} F_{\mu\nu} \tilde{F}_{\mu\nu} = \mathcal{N}_{CS}(-\infty) - \mathcal{N}_{CS}(\infty), \quad (180)$$

is through the spectral flow of the eigenvalues of the Dirac Hamiltonian \mathcal{H} , where \mathcal{H} is defined by

$$i\gamma_\mu D_\mu = i\gamma_0(D_0 + \mathcal{H}). \quad (181)$$

For instance, in the case of a single instanton, one finds (in the gauge $A_0 = 0$) that $A_i(\mathbf{x}, -\infty)$ and $A_i(\mathbf{x}, \infty)$ are gauge equivalent: the spectrum of \mathcal{H} is the same at $t = \pm\infty$. This however does not imply that individual levels $E_j(t)$ are the same at $t = \pm\infty$. In fact, the existence of one right handed zero mode (which is known to be there from the index theorem¹²⁰), implies that precisely one right handed mode crosses zero in going from $t = -\infty$ to $t = \infty$ so that (in adiabatic approximation¹²¹),

$$\begin{aligned} \psi_1(t) &\sim e^{-E_1(-\infty)t}; & t \rightarrow -\infty \\ \psi_1(t) &\sim e^{-E_1(\infty)t}; & t \rightarrow \infty \end{aligned} \quad (182)$$

with $E_1(-\infty) < 0$; $E_1(\infty) > 0$. Otherwise, this mode would not be normalizable in four dimensions.

Furthermore the point t at which the level crossing occurs, is characterized by the fact that the equation $\mathcal{H}\eta = 0$ has for such t a solution, normalizable in *three* space dimensions.

In the case of the valley background, $A_i(\mathbf{x}, -\infty)$ and $A_i(\mathbf{x}, \infty)$ are not only gauge equivalent but corresponds to the same Chern-Simons number (that is to say that the full integrated topological density - the Pontryagin number - is zero). Thus we expect that $E_j(-\infty) = E_j(\infty)$ also for all j . Nevertheless, at least for widely separated instanton anti-instanton pair, we expect from clustering argument a non trivial spectral flow such as in Fig. 13a: a right handed fermion level must cross zero at (more or less) the instanton time position and then cross zero back near the anti-instanton site. In this way the physics of instanton or of anti-instanton (anomalous fermion generation or annihilation) would be locally reproduced by the valley.

If such a picture is confirmed for large R/ρ , then one can ask what happens for small R/ρ . The strategy is thus to check the method of level crossing where physics is well understood, and then use the same tool to explore the unknown territory.

The analysis made by the present authors¹²⁵ (see Appendix C) confirms fully the above picture (Fig. 13a) as long as the instanton anti-instanton distance is large enough: $R/\rho > \sqrt{4/3} \sim 1.1547\dots$

(" $\sqrt{4/3}$ " is only a fit to our numerical data. However, as can be easily seen from Eq.(35) and Eq.(179), the maximum of the variation of the Chern Simons number ($C(0)$) reaches $1/2$ precisely at $R/\rho = \sqrt{4/3}$ ($z = 3$). This makes one suspect that the critical separation is exactly $\sqrt{4/3}$.)

On the other hand, for R/ρ less than the critical value $\sqrt{4/3}$, no level crossing is found to occur. The situation is shown in Fig. 13b. If we identify the level crossing with the (anomalous) chiral fermion generation or annihilation (as is the case in the single instanton or anti-instanton background), then we arrive at the conclusion that the valley field with strongly overlapping instanton anti-instanton configurations ($R/\rho < \sqrt{4/3}$) has nothing to do with anomaly: it is a purely perturbative background. The implication of this on the question of $\Delta(B + L) \neq 0$ cross section in the standard electroweak theory will be discussed in the next subsection.

Furthermore, it was found numerically that at a crossing point t^* (the value of t such that a normalizable solution exists), the relation

$$C(t^*) = \Delta\mathcal{N}_{CS} = \frac{1}{2} \quad (183)$$

always holds. This result is, after all, very natural since $\Delta\mathcal{N}_{CS} = \frac{1}{2}$ corresponds precisely to gauge fields sitting on top of the hill between the two adjacent vacua.

To conclude, the results described in subsection 5.3 and 5.4 imply that the "instanton anti-instanton" valley configuration ceases to be topologically significant when the parameter R/ρ is equal or smaller than the critical value, $\sqrt{4/3}$, or put more intuitively, when the instanton pair "overlaps" substantially. But this means

that the lower portion of the valley has not a well-defined physical meaning: it is just a sort of perturbative field, to be considered together with generic fluctuations around $A_\mu = 0$.

Such a result is not really surprising. In non-Abelian gauge theories the perturbative series is divergent and believed not to be even Borel summable¹³⁵. In other words, the perturbative series alone does not define the theory. This is no reason for despair however: we *know* that in these theories there are also physical, non-perturbative effects to be taken into account. It is to be expected that only the *sum* of perturbative and non-perturbative contributions have a well-defined physical meaning, not each of them separately. The systematic treatment of such a summation and a better definition of the theory, let alone the solution to the unsolved problem of renormalons, are however not known at present.

5.5. *High-energy $\Delta(B + L) \neq 0$ electroweak scattering*

Let us come back to the physics of $\Delta(B + L) \neq 0$ cross sections at high energies. Khoze and Ringwald⁴⁹ have made a simple model calculation via optical theorem, see Section 1.4, in which the valley solution Eq.(34) of the pure $SU(2)$ Yang-Mills theory is used in conjunction with the Higgs contribution in the action, $-\pi\rho^2v^2$. By putting simply a product of the standard zero modes for the external particles (they actually considered a forward elastic amplitude with external gauge bosons, hence these zero modes are replaced by instanton or anti-instanton solution). By performing the integrations over the collective coordinates with the saddle point method, they found that the cross section grew with energy and at an energy of the order of the sphaleron mass, $x = x_{KR} = 8\sqrt{3/5}$ the exponential t'Hooft suppression was overcome completely. See Fig. 4a.

Although it is a toy model calculation (the valley equation used there does not take into account the Higgs coupling to the gauge fields), it is worthwhile to reflect upon its meaning. Since it uses the optical theorem it in principle takes into account the unitarity constraints: how did it manage to evade the "half-suppression" result mentioned earlier?

The answer appears to be that a naive treatment of the external particles has led one astray. Indeed, the saddle point values of R and ρ (which depend on the energy) are such that at x_{KR} , $\rho = 0$; $R/\rho = 0$. See Fig. 4b. The "valley" field with these parameters is simply a perturbative vacuum, $A_\mu = 0$. No wonder the cross section is unsuppressed!

Moreover, the analysis reviewed in the two preceding subsections strongly suggests that the transition to the purely perturbative regime occurs actually much earlier, when the parameter R/ρ reaches $\sqrt{4/3}$. Translated into the value of the valley action, it means that only the "higher" portion of the valley with

$$S > \frac{16\pi^2}{g^2}(0.5960\dots) \quad (184)$$

has anything to do with the $\Delta(B + L) \neq 0$ cross section. Taking into account all

the other terms, we conclude that in this toy-model calculation, the $\Delta(B+L) \neq 0$ cross section is always suppressed at least by

$$\sigma_{\Delta(B+L) \neq 0} \leq e^{-\frac{4\pi}{\alpha}(0.3185\dots)}. \quad (185)$$

Generalizing the lesson learned above, one can give an argument for the "half suppression" result which does neither depend on the particular valley trajectory Eq.(34) nor on its unjustified use, but only on a number of general assumptions. The assumptions needed are the following.

(1) There is a semi-classical Euclidean, real background A_μ (or more properly, an ensemble of them as in the valley trajectory) which dominates the four point function, to be continued and LSZ-amputated to yield the "anomalous" imaginary part of the elastic amplitude.

Although this is just an assumption - quite a nontrivial one -, and it might sound rather naive, after all those discussions about the complex saddle point fields reviewed in Section 2, there is a clear advantage here. As one uses the optical theorem here, all final particles are implicitly already summed over (unitarity!); the only effect of the external particles are four fermion (or bosonic, if one prefers to study purely bosonic amplitude) fields, appearing as pre-exponential factors in the functional integral. It is much less likely here (as compared to the calculation of the *production amplitude*, to be squared and summed over the final states) that the external particles affect the relevant gauge background. We recall also that in the approach by Diakonov and Petrov⁹⁴ (see Section 3), the relevant fields are real and Euclidean (although singular).

(2) This gauge background, to be able to give rise to the "anomalous" imaginary part, must have sufficiently large variations of the Chern-Simons number: more precisely we assume that $C(t)$ crosses the value $1/2$ at least twice (once upwards and once downwards, as required by $Q=0$). Namely, we suppose that

$$\begin{aligned} \mathcal{N}_{CS}(t_1) - \mathcal{N}_{CS}(-\infty) &= \frac{1}{2}, \\ \mathcal{N}_{CS}(t_2) - \mathcal{N}_{CS}(\infty) &= \frac{1}{2}, \end{aligned} \quad (186)$$

($t_1 \leq t_2$), as a (necessary) condition for the fermion level crossings (to and back, as in the valley with $R/\rho > \sqrt{4/3}$) to occur. The action of such a gauge background is easily shown to be at least as large as the single instanton action, $2\pi/\alpha$. Indeed, since $\text{Tr}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu})^2 \geq 0$, for each x , it follows from Eq.(186) and Eq.(177) that

$$\begin{aligned} S &= \frac{1}{2} \int d^4x \text{Tr} F_{\mu\nu}^2 \\ &\geq \frac{1}{2} \int_{-\infty}^{t_1} dt \int d^3x \text{Tr} F_{\mu\nu}^2 + \frac{1}{2} \int_{t_2}^{\infty} dt \int d^3x \text{Tr} F_{\mu\nu}^2 \\ &\geq -\frac{1}{2} \int_{-\infty}^{t_1} dt \int d^3x \text{Tr} F_{\mu\nu} \tilde{F}_{\mu\nu} + \frac{1}{2} \int_{t_2}^{\infty} dt \int d^3x \text{Tr} F_{\mu\nu} \tilde{F}_{\mu\nu} = \frac{2\pi}{\alpha}. \end{aligned} \quad (187)$$

There are also the exponent depending on the initial energy (coming from the external particles through the LSZ procedure), the part of the action due to the Higgs particle, as well as an integration over the collective coordinates. The saddle point equation however normally has a solution in which each term of the exponent is of the same order of magnitude. This leads to the anomalous cross section, suppressed by something like the square root of the t'Hooft factor, in view of Eq.(187).

This argument is admittedly not very rigorous, and should be regarded at best as a tentative one. Nevertheless, it concisely takes into account the two main ingredients of the whole problem, *unitarity* and *anomaly* in an essential manner. Further efforts are welcome to see whether it can be made more rigorous.

5.6. Note on anomaly for massive fermions

All calculations described up to now are done for massless fermions, for simplicity. In the actual world all known fermions (except perhaps for some or all of the neutrinos) are massive. On the one hand, one believes that the physics at high energies, $E \gg m_f$, (m_f standing for a generic fermion mass) should be the same as that with $m_f = 0$. On the other hand it is not at all obvious that anomalous processes such as Eq.(4) do occur with massive fermions. Consider for instance the description of chiral anomaly in terms of the level crossing (recalled at the beginning of subsection 5.4). The smallest (in the absolute value) eigenvalue of the Hamiltonian which crosses zero once - in the case of an instanton background - has the magnitude which is really very close to zero: $E_1(\pm\infty) = O(\frac{1}{L})$, if L denotes the linear size of the spacetime volume in which our system is put. If the fermion is massive, the positive and negative energy levels are separated by a finite gap $2m_f$; how an infinitesimal "level crossing" (from $E_1(-\infty)$ to $E_1(+\infty)$) can cause a jump from e.g. a negative sea level to a positive one?

The resolution of this paradox (due to Krasnikov et al.¹³² and Anselm et al.¹³³; see also Axenides et al.¹³⁴) hinges upon the very way fermions get mass in the Weinberg-Salam theory (the Higgs mechanism). A fermion mass originates from the Yukawa interaction,

$$L_{Yukawa}^{(up)} = g \epsilon \bar{\psi}_L \phi \psi_R + h.c. \quad (188)$$

if the Higgs field has a constant part, $\phi = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} + \dots$

Now the (constrained) instanton responsible for anomalous process Eq.(4) behaves as Eq.(12) near the instanton center. From Eq.(12) it follows that the fermion mass term vanishes at the instanton center, i.e. precisely where the eigenvalue of the Dirac Hamiltonian changes sign. For this reason the massive fermion experiences the level crossing just as a massless one. "Massive zero modes" have been constructed, and an analogous paradox is solved in the case of the global SU(2) anomaly.¹³⁴

6. Conclusion

A unified physical picture seems to emerge from various different types of analyses reviewed here. A single instanton in the Euclidean formulation describes the vacuum-to-vacuum tunnelling and, as such, is not directly related to the $\Delta(B+L) \neq 0$ electroweak transition in TeV-energy scattering processes. The overlap of a high-energy two-particle state with most likely multiparticle states of the same energy and quantum numbers *within the same well*, enters as a multiplicative factor in the amplitude. The rest of the amplitude describes the tunnelling between states of adjacent "wells", whose rate rapidly grows with energy, and loses the exponential suppression altogether as the energy exceeds the barrier height, the sphaleron energy.

The first factor, which describes the mismatch between the "most favorable" state (for the purpose of barrier penetration) and the initial two-particle state, gets instead strongly suppressed as the energy grows, as has been elegantly illustrated in several quantum mechanical analogue problems, reported in Section 3. It is indeed an analogue of the Landau's semi-classical matrix elements⁸⁶.

It has at high energies the same semi-classical form, $\exp -\frac{1}{\alpha}$ as the familiar tunnelling factor, although it describes a transition within the same well. Ultimately, this factor seems to be responsible for the exponential suppression of the $\Delta(B+L) \neq 0$ cross section at high energies found in different analyses. The same factor is found also in the unitarized multi-instanton approximation (compare Eq.(137) with Eq.(151)).

In the instanton description, the energetic initial particles must first convert themselves into $O(\frac{1}{\alpha})$ gauge-bosons, Higgs, plus the original fermions, so that these particles, having small energies, can be absorbed by the instanton without any form-factor suppression. The price paid for the conversion, $\sim (\alpha)^{1/\alpha}$, gives rise to the mismatch factor⁶³.

The same Landau factor seems to be at the origin of the qualitative difference between the $\Delta(B+L) \neq 0$ transition at high temperature or at high fermion density where the transition rate becomes eventually unsuppressed, and the high energy scattering in which baryon number violation is most likely to remain unobservable.

All in all, consistency of the result (at least "half-suppression" of $\Delta(B+L) \neq 0$ cross sections) found from a variety of different kinds of analysis (Section 3, Section 4 and Section 5) seems to be quite remarkable. This, together with its physical understanding (in terms of the Landau factor as well as in terms of the general unitarity bound), provides us with certain degree of confidence in the result.

To be scrupulous, the final outcome of new, semi-classical approaches (reviewed in Sections 2 and 3) is not known as yet. However, *if* these analyses eventually indicated an unsuppressed $\Delta(B+L) \neq 0$ cross sections at the sphaleron energy, it would come as a surprise. It would be a new (and very interesting) phenomenon, which has however very little to do with the original sphaleron or instanton calculations, and for which no hints exist for the moment. But, of course, who knows?

Barring such a possibility, then, have we made just "MUCH ADO FOR NOTHING", after all?

There are reasons to believe that this is too pessimistic a viewpoint. The original suggestion^{20,21} that the instanton cross section is corrected by an exponentially growing factor at *low energies* is now well established. If the enhancement is so as to convert the 't Hooft factor into something close to its square root (an optimist would call it "half-enhancement"),

$$e^{-\frac{4\pi}{\alpha}} \longrightarrow e^{-\frac{2\pi}{\alpha}}, \quad (189)$$

this will be insignificant in the electroweak theory, but might be rather important in the low-energy instanton physics in QCD. In spite of the generally accepted belief that the instantons play an important role in the physics of low-lying pseudoscalar mesons (for instance, in the solution of the "U(1) problem"¹³⁶), quantitative understanding of the role of instantons in these systems has not yet been achieved. Any improved understanding of physics related to instantons should be helpful. The same can be said, with an even stronger emphasis, with regard to the instanton liquid model of the QCD vacuum¹²³.

Finally, all these issues are deeply related to the large-order breakdown of perturbation theory and connected problems¹³⁷, especially in QCD^{96,138}. In our opinion no truly significant step forward has been made after the 't Hooft's work¹³⁵ in this regard. It is hoped that somewhat improved understanding of instanton physics achieved through the study of baryon number violation in high-energy electroweak interactions as reviewed here, will help clarifying these issues as well in a near future.

Appendix A Saddle-point evaluation of multi instanton sizes

The integral over t_i , in Eq.(133-135) can be easily evaluated since it is factorized: it leads to the substitution,

$$-iEt_i + \left(\frac{96\pi^2}{g^2}\right)\rho_i^2\rho_{i+1}^2/t_i^4 \longrightarrow 5\left(\frac{96\pi^2}{g^2}\rho_i^2\rho_{i+1}^2\right)^{1/5} \quad (i = 1, 2, \dots, 2n-1) \quad (A.1)$$

in the exponent, Eq.(134). The final integrations over the instanton sizes have the form,

$$\int \int \dots \int \prod_{i=1}^{2n} d\rho_i \text{ (powers of } \rho' \text{'s) exp}(\bar{W}) \quad (A.2)$$

where

$$\bar{W} \equiv -\pi^2 v^2 \sum_{i=1}^{2n} \rho^2 + 5 \sum_{i=1}^{2n-1} \left(\frac{96\pi^2}{g^2}\rho_i^2\rho_{i+1}^2\right)^{1/5}. \quad (A.3)$$

For the purpose of performing the above integration by the saddle point method one can introduce the dimensionless variables z_i instead of ρ 's,

$$\rho_i = \left(\left(\frac{96\pi^2}{g^2}\right)E^4\pi^{10}v^{10}\right)^{-1/6} z_i^{5/2}, \quad (A.4)$$

in terms of which the exponent becomes

$$\bar{W} = \left(\frac{96E^4}{\pi^2 g^2 v^4} \right) Y, \quad (\text{A.5})$$

$$Y \equiv - \sum_{i=1}^{2n} z_i^5 + \left(\frac{5}{4^{4/5}} \right) \sum_{i=1}^{2n-1} z_i z_{i+1}. \quad (\text{A.6})$$

The saddle point equations for z_i , though simple, do not appear to be solvable exactly for generic value of n . However for n sufficiently large (in practice it means $n \geq 6$: see below) the solution can be found easily numerically. One finds

$$z_i = z_{2n-i+1}, \quad (i = 1, 2, \dots, n), \quad (\text{A.7})$$

and

$$\begin{aligned} z_n &\simeq z_* = 2^{-1/5} = 0.87055056\dots; \quad n \geq 6, \\ z_1 &= 0.72808; \quad z_2 = 0.85185; \quad z_3 = 0.86817; \\ z_4 &= 0.87025; \quad z_5 = 0.87051; \quad z_6 = 0.87055 \simeq z_* \end{aligned} \quad (\text{A.8})$$

Inserting the above solution in Eq.(A.6) one finds

$$Y \simeq \frac{3}{2}n - 1.06386. \quad (\text{A.9})$$

The fact that the first and last few z_i 's differ from z_* - an end point effect - is responsible for the second term of Eq.(A.9) and is crucial for the subsequent discussions. Substituting Eq.(A.9) into Eq.(A.5) one finds Eq.(136).

Appendix B Proof of absence of the fermion zero modes in the valley

The proof of the absence of the zero modes of the Dirac operator $\mathcal{D} \equiv i\gamma_\mu(\partial_\mu - igA_\mu)$ in the valley background is straightforward. The gauge field is given by Eq.(34) which can be written as

$$A_\mu = -\frac{i}{g}(\sigma_\mu \bar{\sigma}_\nu - \delta_{\mu\nu}) F_\nu \equiv \bar{\eta}_{\mu\nu}^a F_\nu \sigma^a \quad (\text{B.1})$$

$$F_\mu = \frac{1}{2} \partial_\mu \log L(x) \quad L(x) \equiv \frac{(x - x_a)^2 + \rho^2}{(x - x_i)^2 + \rho^2} (x - x_i + y)^2. \quad (\text{B.2})$$

We consider the decomposition:

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}. \quad (\text{B.3})$$

Because the valley is a superposition of an instanton (with a righthanded zero mode) and an anti-instanton field, (with a lefthanded one), the only physical possibility (to be ruled out) is that *both* \mathcal{D}_+ and \mathcal{D}_- have a single, non degenerate, zero mode.

Using the explicit form of η^1), one has

$$\mathcal{D}_+ \equiv -\partial_t + i\vec{\sigma} \cdot \vec{\nabla} + i\vec{\tau} \cdot \vec{F} - \vec{\sigma} \wedge \vec{\tau} \cdot \vec{F} - F_0 \vec{\sigma} \cdot \vec{\tau} \quad (\text{B.4})$$

$$\mathcal{D}_- \equiv +\partial_t + i\vec{\sigma} \cdot \vec{\nabla} - i\vec{\tau} \cdot \vec{F} - \vec{\sigma} \wedge \vec{\tau} \cdot \vec{F} - F_0 \vec{\sigma} \cdot \vec{\tau} \quad (\text{B.5})$$

where $x_\mu \equiv (t, \vec{r})$,

$$\vec{F} = \vec{r}f(r, t); \quad F_0 = F_0(r, t) \quad (\text{B.6})$$

$r = \sqrt{\vec{r}^2}$, $\vec{\sigma}$ are spin operators, and $\vec{\tau}$ are isospin one.

Since \mathcal{D} commutes with the three dimensional total angular momentum operator $\vec{J} = -i\vec{r} \wedge \vec{\nabla} + \vec{\sigma} + \vec{\tau}$, the requirement of having a nondegenerate zero mode is satisfied if $j = 0$.

Let us concentrate on the left-handed mode, associated with \mathcal{D}_+ below. The most general function with $j = 0$ can be written accordingly¹¹⁵ as:

$$\eta_{j\alpha} = (\sigma_2)_{j\alpha} S(r, t) - i(\vec{\sigma}\sigma_2)_{j\alpha} \cdot \vec{r}T(r, t) \quad (\text{B.7})$$

The equation $\mathcal{D}_+\eta = 0$ is equivalent to:

$$\partial_t \tilde{S} - h \partial_r \tilde{T} = 0; \quad -\partial_r \tilde{S} - h \partial_t \tilde{T} = 0, \quad (\text{B.8})$$

where $h = h(r, t) = \frac{L(x)^2}{r^2}$ and two new functions \tilde{T} and \tilde{S} ,

$$T = \frac{1}{r^3} L^{\frac{1}{2}} \tilde{T}; \quad S = L^{-\frac{3}{2}} \tilde{S}, \quad (\text{B.9})$$

have been introduced. The consistency condition following from the above is equivalent to $\nabla(h\nabla\tilde{T}) = 0$ (here $\nabla \equiv (\partial_r, \partial_t)$). Furthermore the normalization condition reads

$$\begin{aligned} \int d^4x r^2 |T|^2 &= \int_{-\infty}^{+\infty} dt \int_0^{+\infty} dr 4\pi \frac{L}{r^2} \tilde{T}^2 < \infty; \\ \int d^4x |S|^2 &= \int_{-\infty}^{+\infty} dt \int_0^{+\infty} dr 4\pi r^2 L^{-3} \tilde{S}^2 < \infty. \end{aligned} \quad (\text{B.10})$$

Note that with our choices of $x_i = (-\frac{R}{2}, 0, 0, 0)$ and $x_a = (\frac{R}{2}, 0, 0, 0)$, $L(r, t) \neq 0$ almost everywhere on the line $r = 0$, so that a necessary condition for normalization is $\tilde{T}(0, t) = 0$.

One is thus led to the boundary value problem:

$$\begin{aligned} \nabla(h\nabla\tilde{T}) &= 0, & \text{on } \Omega; \\ \tilde{T} &= 0, & \text{on } \partial\Omega; \\ \tilde{T} &\in H^2(\Omega), & \tilde{T} \in C^0(\bar{\Omega}), \end{aligned} \quad (\text{B.11})$$

where $\Omega \equiv \{(r, t) \in \mathbf{R}^2 / r > 0\}$.

The operator $\nabla^2 + \nabla h \nabla$ is an elliptic differential operator with coefficients analytic in Ω . At this point one can use standard theorems to prove that \tilde{T} cannot have global extrema on Ω . Using the continuity on $\bar{\Omega}$, the boundary conditions Eq.(B.10) and the condition $\tilde{T} \rightarrow 0$, $r, t \rightarrow \infty$, it is easy to prove that $\tilde{T} = 0$.

It follows that \tilde{S} must be a constant, i.e.,

$$S = \text{const. } L(x)^{-\frac{3}{2}}.$$

The normalization condition however forces $S = 0$. *Q.E.D.*

Having disposed of the zero modes, one might then ask if the zero mode of \mathcal{D}_+ in the antistanton field, and that of \mathcal{D}_- in the instanton background, combine somehow in the valley field so as to form a pair of non zero eigenvalue $\pm\lambda$ of \mathcal{D} . A theorem by Ikebe and Uchiyama¹¹⁸ however excludes this possibility.

Appendix C Level-crossing in the valley

The Dirac Hamiltonian \mathcal{H} is defined by: $i\gamma_\mu D_\mu = i\gamma_0(D_0 + \mathcal{H})$.

Decomposing

$$\mathcal{H} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix},$$

and passing to a noncovariant formalism, one finds:

$$H_+ = -H_- = H, \tag{C.1}$$

with

$$H(t) = +i\vec{\sigma} \cdot \vec{\nabla} - \vec{\sigma} \wedge \vec{\tau} \cdot \vec{F} - F_0 \vec{\sigma} \cdot \vec{\tau} \tag{C.2}$$

where the notation is the same as in Appendix B, Eq.(B.6).

To simplify the problem, one can just look for zero modes of $\mathcal{H}(t; R)$ without studying the detailed behavior of levels in terms of parameters t, R .

The condition (C.1) implies that zero modes always appear in pairs of opposite chirality. We assume that for each chirality zero modes are non degenerate, so that we must look for singlets of the total spatial angular momentum (which is a symmetry of \mathcal{H}). The most general form of the singlet η is given in (B.7).

The equation $H\eta = 0$ then reads:

$$\begin{aligned} 3F_0 S - 3T - \vec{r} \vec{\nabla} T + 2\vec{F} \vec{r} T &= 0, \\ -F_0 r^2 T + \vec{r} \vec{\nabla} S + 2\vec{F} S &= 0. \end{aligned} \tag{C.3}$$

Making the substitution

$$S = \frac{1}{L} \tilde{S}, \quad T = \frac{L}{r^3} \tilde{T}, \tag{C.4}$$

we get a simplified system:

$$\tilde{T}'(r; t) = 3F_0 \frac{r^2}{L^2} \tilde{S}; \quad \tilde{S}'(r; t) = F_0 \frac{L^2}{r^2} \tilde{T}, \tag{C.5}$$

where the prime means differentiation with respect to r and the time t appears as a parameter.

Normalizability of the solution implies:

$$\infty > \int_0^\infty dr r^2 |S|^2 = \int_0^\infty dr \frac{r^2}{L^2} |\tilde{S}|^2; \quad \infty > \int_0^\infty dr r^4 |T|^2 = \int_0^\infty dr \frac{L^2}{r^2} |\tilde{T}|^2. \quad (\text{C.6})$$

We wish to know for which range of the parameter R and for which values of t - if any - (the other parameter ρ just fixes the scale) the system Eq.(C.5) has a normalizable solution. Normalizability enforces the following initial condition (at a given t)

$$\tilde{S}(0) = 1; \quad \tilde{T}(0) = 0; \quad \text{if } t \neq y + \frac{R}{2} \quad (\text{C.7})$$

$$\tilde{S}(0) = 0; \quad \tilde{T}(0) = 1; \quad \text{if } t = y + \frac{R}{2}, \quad (\text{C.8})$$

where y is defined (35).

The problem turns out to be too hard to be treated analytically: it suffices to note that the system is equivalent to a second order linear equation for \tilde{T} with 12 regular fuchsian points! In passing we recall¹²⁸ that for the case of a pure anti-instanton (or an instanton) a normalizable solution can be found analytically. Lacking a better method a numerical method was used to solve Eq.(C.5). The strategy adopted is as follows.

- (i) First solve the system by a power series in r at $r \simeq 0$ and $r \rightarrow \infty$. It turns out that in each region one and only one of the two independent solutions is compatible with the normalizability.
- (ii) Choose, for a given t and R/ρ , the normalizable solution for \tilde{T}, \tilde{S} near $t = 0$. Make them evolve according to Eq.(C.5) up to a very large but fixed value of r , r_Λ .
- (iii) Plot $\tilde{T}(r_\Lambda)$ as a function of t , for a fixed R/ρ . See whether $\tilde{T}(r_\Lambda)$ crosses zero as t is varied. If it does at some t , it means by continuity that for t very close to that value there is a normalizable solution. The power series solution of the coupled equation assures that whenever \tilde{T} is normalizable, so is \tilde{S} .
- (iv) Repeat the same procedure for different values of R/ρ .

Note that this method is somewhat similar to that used in the proof of the so-called oscillation theorem in one-dimensional quantum mechanics¹²⁹. It is also reminiscent of the "shooting method" used by mathematicians for solving certain differential equations.

The result cited earlier (Fig. 13a and Fig. 13b) was obtained this way: for all values of R/ρ above a critical value,

$$\frac{R_c}{\rho} \simeq 1.15470.... \simeq \sqrt{\frac{4}{3}}, \quad (\text{C.9})$$

there are two values of t for which a normalizable solution of $H\eta = 0$ exists. Below the critical separation, no level crossing occurs.

Acknowledgements

Part of this work was done during a visit of the authors to CERN. We thank the members of its Theory Division for hospitality. We thank A. Ringwald and H. Suzuki for useful discussions. One of the authors (K.K.) wishes to thank the Physics Department of University of Pisa for hospitality.

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Figure captions

Fig 1. Graphs representing: a) the leading approximation, b) a soft-soft correction, c) a hard-soft corrections, d) a hard-hard correction, in a theory with fermions

(in an instanton background).

Fig 2. The leading contribution to the exponent in the R -term approach: double residue of W classical field integrated over the phase space.

Fig 3. The action of the instanton anti-instanton valley (in pure gauge theory), plotted in terms of R/ρ .

Fig 4. a) The saddle point exponent, b) the saddle point ratio R/ρ , versus the parameter $x = E/E_{sp}$ in the Khoze-Ringwald model.

Fig 5. The lowest hard-hard correction in a pure gauge theory.

Fig 6. Example of "squared tree" multiloop graph in the instanton background (blobs means tree graphs).

Fig 7. The relevant complex-time path in the multiparticle approach.

Fig 8. The behavior of the singular solution used in the W.K.B. approach (B and C are to be intended located at $-\infty$).

Fig 9. The contours γ_1 and γ_2 in the complex q plane, used in the W.K.B. approach.

Fig 10. A multiinstanton chain. The solid lines indicate exchange of many bosons, the lines with an arrow representing fermions. For simplicity of drawing N_F is taken to be 2 here.

Fig 11. $C(x_4)$ versus x_4/ρ for $\frac{R}{\rho} = 10$ (outmost curve), $\frac{R}{\rho} = 5$, $\frac{R}{\rho} = 2$ (middle), $\frac{R}{\rho} = 1$ and $\frac{R}{\rho} = 0.5$ (innermost curve).

Fig 12. The maximum of $C(x_4)$, $C(0)$, as a function of $\frac{R}{\rho}$.

Fig 13. Schematic behavior of the level crossing for the Dirac Hamiltonian in the valley background, a) for $R > R_c$, b) for $R < R_c$.

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